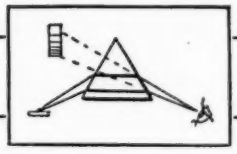
Journal of the OPTICAL SOCIETY *of* AMERICA

VOLUME 27
NO. 2



FEBRUARY
1937

New Method for the Determination of the Transmission Factor of a Spectroradiometer . . . . . K. K. KAY AND H. M. BARRETT 65

Infrared Radiation and Visual Function . . . . . MATTHEW LUCKIESH AND FRANK MOSS 69

Effect of Irradiation on the Transmission of Lithium Fluoride . . . . . EDWIN G. SCHNEIDER 72

Correction to "Changes in Color Temperature of Tungsten-Filament Lamps at Constant Voltage" . . . . . DEANE B. JUDD 74

Hamilton's Method in Geometrical Optics . . . . . J. L. SYNGE 75

Characteristics of Some New Mercury Arc Lamps . . . . . B. T. BARNES AND W. E. FORSYTHE 83

Optical Changes in Freshly Evaporated Aluminum-Magnesium Films . . . . . HIRAM W. EDWARDS AND ROBERT P. PETERSEN 87

Absence of Fresnel's "Most Useful Effect" in the Hyperbolic Reflector . . . . . L. E. DODD 92

Polarization Phenomenon in the Hilger E_1 Quartz Spectrograph . . . . . JOSEPH W. ELLIS AND JOSEPH KAPLAN 94

Published monthly
at Prince and Lemon Sts., Lancaster, Pa.
by the
AMERICAN INSTITUTE OF PHYSICS
Incorporated
175 Fifth Avenue, New York, N. Y.
FOR AND UNDER THE EDITORIAL SUPERVISION OF
THE OPTICAL SOCIETY OF AMERICA

Journal of the
OPTICAL SOCIETY
of AMERICA

VOLUME 27 FEBRUARY, 1937 NUMBER 2

A New Method for the Determination of the Transmission Factor of a Spectroradiometer

K. K. KAY AND H. M. BARRETT
Department of Physiological Hygiene, University of Toronto, Toronto, Canada
(Received November 3, 1936)

IN connection with atmospheric pollution studies being carried out in this laboratory, it was found necessary to assay the radiant flux of a quartz Hg vapor arc lamp in absolute energy units, interest being particularly centered in the region λ2967–3024A. The spectroradiometer constructed for this purpose is shown in Fig. 1. It is essentially the same as that described by Marshall and Knudson.[1] The optical system consists of a 60° fused silica prism of 60×60×60 mm face, and two fused silica lenses f: 20 cm for the sodium D line. The prism was set at minimum deviation for λ3024A.

The thermopile was constructed after the design of Coblentz[2] and consisted of 22 bismuth-silver junctions in series, mounted on an ivory frame. In soldering the junctions a binocular dissecting microscope of 25×magnification was found to be of great assistance. The thermopile was mounted in a heavy brass box fitted with an adjustable slit. The galvanometer used was a Leeds and Northrup moving-coil type No. 2284 B with resistance of 18.3 ohms. Its sensitivity equalled 0.0151×10^{-6} volt per mm with the scale placed at 3 meters and with the thermopile (resistance = 20.2 ohms) in series. The sensitivity was checked at each reading by means of a potentiometer in series with the galvanometer and thermopile. The sensitivity of the thermopile as determined by means of a National Bureau of Standards carbon lamp No. C–221 equalled 0.197 volt per watt per 22.5 mm² of receiving surface (the surface exposed using a thermopile slit width of 0.90 mm).

The spectroscope including the thermopile was covered completely by a hinged wooden box painted black on the inside. To minimize stray radiation two stops of calculated size were placed in the collimator tube. The collimator slit was fitted with stainless steel jaws with a curvature of 17 cm radius, in order to produce a straight image in the focal plane. The whole instrument was mounted on a heavy iron stand and could be tilted and turned in any desired direction.

STRAY RADIATION

As has been pointed out the present instrument was designed primarily for measuring the intensity of the region λ2967–3024A and it was necessary to use relatively large distances between the lamp and the spectroradiometer slit. Large slit widths were therefore employed in order to obtain as large galvanometer deflections as possible. Fig. 2 shows photographs of λ2967–3024A taken by means of a small Hilger spectro-

[1] A. Marshall, and A. Knudson, J. Am. Chem. Soc. **52**, 2304 (1930).
[2] W. W. Coblentz, Bull. Nat. Bur. Stand. **9**, 1, March (1913).

BOSTON UNIVERSITY
COLLEGE OF LIBERAL ARTS
LIBRARY

graph with a slit width of 1.0 mm and exposure times as indicated. Comparison of the intensities of λ2967–3024A at 0.04 second with the intensity of the stray radiation present at the 1.0 second exposure shows that the stray radiation is considerably less than 4 percent. A similar comparison of the 0.2 second exposure and the 10 second exposure indicates the stray radiation to be probably less than 2 percent.

The Optical Transmission Factor

The spectral transmission of the optical system was determined by a method similar to that used by Forsythe and Barnes.[3] A Hg vapor arc lamp and auxiliary spectroscope set on a movable carriage, acted as the source of monochromatic radiation. A second thermopile was mounted parallel with and to one side of the collimator slit of the spectroradiometer. Comparison of the intensity of the monochromatic beam before and after its passage through the spectroradiometer gave the transmission value for the wave-length used. It is important to mention here that the focal lengths and aperture of the lenses in the auxiliary spectroscope were such that all of the monochromatic radiation falling on the front slit of the spectroradiometer was incident upon the collimator lens. The values obtained for the spectral transmission are shown graphically in Fig. 3, each point on the graph representing an individual determination.

Fig. 1. The spectroradiometer set in a tilted position and enclosing box removed.

Effect of Lamp Distance on the Spectroradiometer Transmission

As is generally known, to use a spectroradiometer to evaluate the radiant flux of an energy source in absolute units, it is essential that all the radiation incident on the slit be incident on the collimator lens. This condition is fulfilled only when a certain minimum distance between the source and the collimator slit is exceeded, this distance depending on the size of the source, the slit width, and the focal length and aperture of the collimator lens. In evaluating the intensity of the shorter ultraviolet wavelengths, because of their relatively low intensity, it is usually necessary to place the source at a distance where the above-mentioned condition is not fulfilled. For this reason a second factor must be determined. Anderson, Fraser and Bird[4] have described a method for determining this factor, which involves the use of filters. The present method is believed to be simpler and more direct.

The Hg vapor arc was mounted on an optical bench before the spectroradiometer. The intensities of λ2967–3024A, λ3132A, and λ3663A were determined at distances of from 3 to 66.7 inches. By using the lamp intensity at 33.2 inches as a basis, the intensities at other distances were calculated by means of the inverse square law. Fig. 4 shows the experimental and calculated relationship between lamp distance and intensity for the regions λ2967–3024A, λ3132A, and λ3663A. The salient feature of the curves is that

[3] W. E. Forsythe and B. T. Barnes, Rev. Sci. Inst. **10**, 569 (1930).

[4] W. T. Anderson, Jr., H. T. Fraser, and L. F. Bird, J. Opt. Soc. Am. and Rev. Sci. Inst. **17**, 454 (1928).

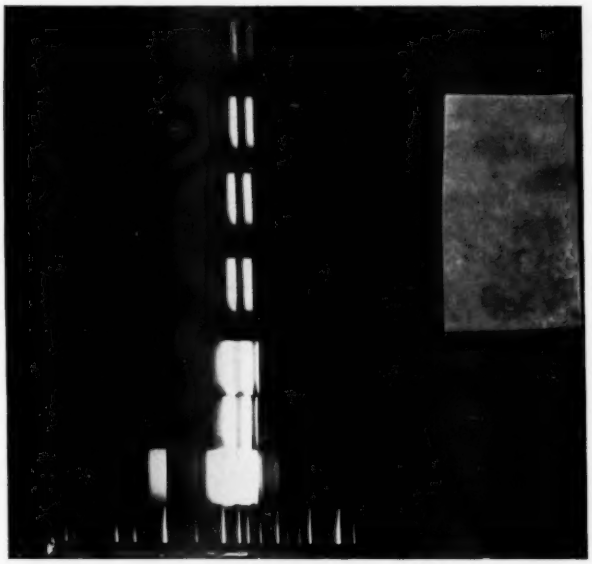


FIG. 2. Photographs of 2967–3024A with exposures as indicated, taken with a small Hilger spectrograph. Spectroradiometer slit width 0.50 mm. Hilger spectrograph slit width 1.0 mm.

the calculated and observed intensity values in the three spectral regions all commence to diverge approximately at the same point and, as shown in Table I, at any given distance the percent divergence is the same for the three regions studied, within the limit of error of the experiment, i.e., ±5 percent. The correction factors to be applied to the observed intensity values at each distance are shown in the last column of the table.

Thus by measuring the intensity of any strong region in the spectrum of a source at a number of distances from the spectroradiometer, the correction to be applied at any distance can be calculated, and since this factor is independent of wave-length it can be applied to intensity measurements in any other region.

It might be mentioned at this point that as the lamp distance is increased beyond the point where the aperture of the collimator lens is just filled by the radiation from the source, the thickness of fused silica traversed by the radiation will vary due to the convergence of the beam. This fact, particularly with lenses of small curvature, might lower the optical transmission, especially for the shorter wave-lengths. With the present instrument this effect was negligible as is shown by the fact that the calculated and observed

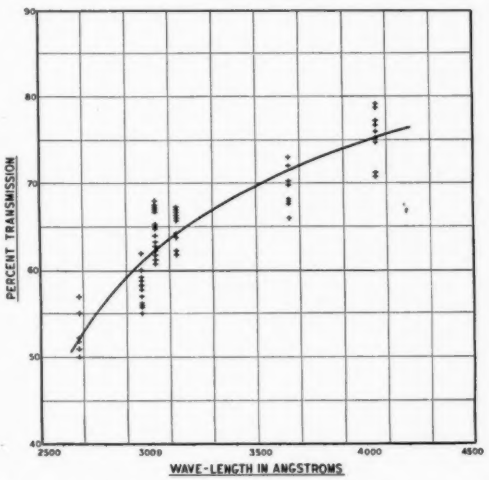


FIG. 3. Spectral transmission of the optical system of the spectroradiometer.

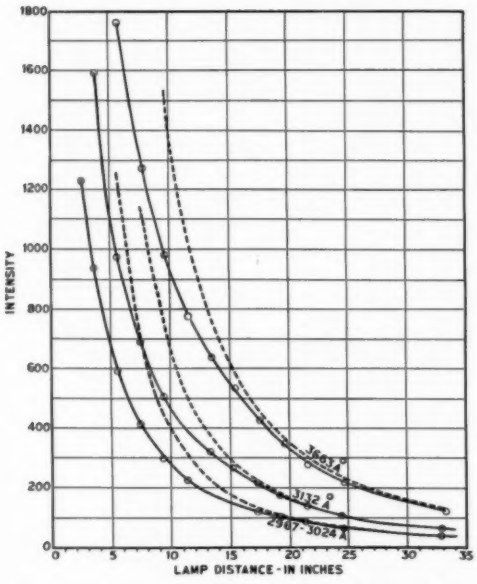


FIG. 4. Full line curves show observed intensities of a Hg vapor arc lamp at various distances. Broken curves show intensities calculated by means of the inverse square law using intensities at 33½ inches as basis. Indicates loss of intensity at distances closer than 17 inches.

intensity values for the wave-lengths studied agree at distances very considerably beyond the point where this factor could come into play.

TABLE I. *Percent difference between experimental and calculated intensity curves shown in Fig. 4.*

LAMP DISTANCES (INCHES)	λ2967–3024A	λ3132A	λ3663A	AVERAGE $=a$	DISTANCE TRANSMISSION FACTOR $=1/(100-a)$
66.7		−3	−6		
49.9	−6	0	0		
33.2	0	0	0	0	1.00
24.7	−2	3	3	1	1.01
21.7	1	5	7	4	1.04
19.7	5	3	3	4	1.04
17.7	8	5	5	6	1.06
15.7	11	8	8	9	1.10
13.7	17	10	15	14	1.2
11.7	24	21	25	23	1.3
9.7	31	30	34	32	1.5
7.7	39	41	47	42	1.7
5.7	52	56	59	56	2.3
3.7	68	69	71	69	3.2
2.7	78	78	81	79	4.8

To summarize, in standardizing an ultraviolet lamp by this method two corrections are to be applied to the observed intensity values: first, the spectral transmission factor which corrects for the absorption in the optical system of the spectroradiometer, and secondly, the distance transmission factor, determined by the method outlined above, which takes into account that fraction of radiation incident on the slit, which impinges on the walls of the collimator tube and therefore does not pass through the instrument to the thermopile.

Infrared Radiation and Visual Function

MATTHEW LUCKIESH AND FRANK K. MOSS
Lighting Research Laboratory, General Electric Company, Nela Park, Cleveland, Ohio
(Received December 4, 1936)

FOR many years infrared radiation has been under suspicion as a possible cause of certain ophthalmic effects. This is evidenced by the production and use of certain eyeglasses for special, and even everyday use, due chiefly to assumptions or considerations of Crookes and others. Notwithstanding this interest, infrared radiation has not been convicted by direct evidence excepting in cases where the heating effect has been obvious and excessive. However, in addition to this old avenue of interest, another has been reopened recently.

Researches demonstrating the value of light and lighting in the performance of exacting visual tasks and the availability of efficient light sources and lighting equipments have combined to increase prevailing levels of illumination as well as those which are now recommended as desirable. These developments bring into prominence, at least from a theoretical viewpoint, the possibility of deleterious effects which may result from the absorption of radiant energy in the eye-media.[1] Three major effects, physiological, thermal and pathological,[2] may be produced by radiation. Pathological effects are not considered in the present discussion since (1) infrared radiation apparently has no specific action upon the tissues analogous to that of abiotic or ultraviolet radiation and (2) by reason of the fact that pathological consequences do not result from the use of levels of illumination of the order of magnitude of those involved in the present research.

It is well known that a lumen of daylight has much less total radiant energy associated with it than a lumen of common artificial light. The relative total amounts of radiant energy reflected from common printed pages (so-called white paper and black ink) are shown in Fig. 1 for daylight and for light from various tungsten-filament lamps. Other artificial illuminants can be directly compared by considering their luminous efficiencies.[3] It will be noted that the total reflected energy from a given area of printed matter on white paper is about ten times greater per footcandle for tungsten-filament light than for average daylight. This means, for example, that the total energy reflected toward the eye from a given area of white paper is approximately the same for 100 footcandles of tungsten-filament light as for 1000 footcandles of daylight.

The experimental conditions of the present esearch involved reading for one hour under (1) unfiltered illumination of 100 footcandles from 150-watt tungsten-filament lamps and (2)

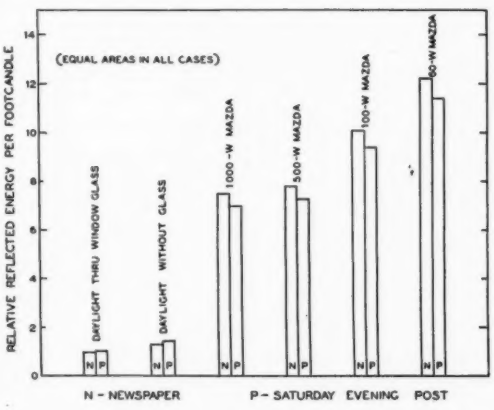


FIG. 1.

[1] "Infrared Radiant Energy and the Eye," M. Luckiesh, Am. J. Physiol. **50**, 383 (1919).

[2] "An Analysis of the Literature Concerning the Dependency of Visual Functions Upon Illumination Intensity," L. T. Troland, Trans. I.E.S. **26**, 107 (1931); "The Effect of Modern Illuminants on the Eyes," G. S. Crampton, Trans. I.E.S. **19**, 96 (1924); "The Eye as Affected by Illumination," LeG. H. Hardy, Trans. I.E.S. 29, 364 (1934); "Irradiation Cataract," J. G. Milner, British J. Ophthal. **18**, 497 (1934); "Pathological Action of Light on the Eye," W. S. Duke-Elder, Lancet **2**, 16 (1926); Physiol. Abstracts **11**, 442 (1926–27), Abs. No. 2656; "The Pathological Effects of Radiation on the Eye," L. Bell and F. H. Verhoeff, Trans. I.E.S. **16**, 625 (1921); "The Physiological Effects of Radiant Energy, Especially upon the Human Eye," C. Sheard, Am. J. Physiol. Optics 5, 214 (1924); "Study of Sun Glare in Iraq," P. C. Livingston, Brit. J. Ophthal. **16**, 577 (1932).

[3] "Spectral Radiant Intensities of Some Tungsten Filament Incandescent Lamps," B. T. Barnes and W. E. Forsythe, J. O. S. A. **26**, 313 (1936).

TABLE I. *These data, with the exception noted, are averages of the results from five separate tests made by each of five subjects possessing normal or near-normal vision. The probable errors of these data, excluding the systematic differences among the subjects, are less than 3 percent.*

CRITERIA	PERIOD	WITHOUT FILTER	WITH FILTER
Number of blinks	First 5 min.	24	23
	Last 5 min.	33	33
Diameter of pupil (millimeters)	Before	2.57	2.50
	After	2.70	2.64
Convergence reserve (prism diopters)	Before	12.0	12.0
	After	11.5	11.4
Time to perform Luckiesh-Moss demonstration visual test* (minutes)	Before	1.19	1.16
	After	1.18	1.14
Rate of reading (page per hour)		29.6	29.9

* Two tests by four subjects

under the same intensity of illumination from the same lamps, but obtained by filtering the light through three inches of water. The absorption of radiant energy, as measured with a thermopile, was approximately 74 percent. In other words, the filter reduced the total energy reflected from the printed matter to about one-fourth of its value without the filter. The results are summarized in Table I for a number of criteria used.

DISCUSSION OF RESULTS

Frequency of eyelid reflex

One of the possible effects of infrared radiation is that of drying the cornea or conjunctiva. If this factor is an important one, it seems reasonable to conclude that the rate of blinking would be higher under the unfiltered light than under the filtered light. However, it will be noted from Table I that the rate of blinking was not appreciably altered as a result of the filter which absorbed a large part of the total radiant energy. Although ambient temperature and relative humidity were not controlled directly during this experiment, a statistical analysis of measurements taken during each test-period indicated that the average values of these variables were approximately equal for the two experimental conditions.

It has been stated that mental tension and rate of blinking are closely related and that the latter may constitute a relief mechanism whereby nervous energy, otherwise unutilized, passes into a highly facilitated path.[4] If this hypothesis is correct, it may be assumed that the reflex blink served not only as a criterion of the thermal effects of infrared radiation, but also as a means for indicating nervous tension developed as a result of reading. Since the rate of blinking was about the same in both cases, it may be concluded that the task of reading was apparently equally difficult under the filtered and unfiltered lights.

This conclusion is further supported by the results of a recent research which we directed towards ascertaining the appropriateness of the reflex blink as a criterion of visual conditions unfavorable for easy seeing.[5] In this research, the rate of blinking was observed under the following experimental conditions: (1) during the first and last five minutes, respectively, of an hour of continuous reading, (2) during reading of types of different sizes, (3) during the performance of a visual task requiring rapid and alternate fixation of two different test-objects, (4) during reading with and without a glare-source within the visual field and (5) during reading with accurate and inaccurate ophthalmic corrections by means of eyeglasses. In all cases, when a change was made from one condition to a less favorable one for seeing, the rate of blinking increased significantly. Hence it may be concluded in the present discussion that a significant benefit is not obtained by decreasing the total radiant energy from tungsten-filament lamps by water filters since the rate of blinking is substantially the same in both cases.

Size of the pupil

It is a matter of common experience that the pupil is constricted if the eye is irritated in any way. It also has been demonstrated that the pupil is dilated, to some extent, as a result of fatigue induced by protracted periods of exacting visual effort.[6] Therefore, it may be assumed for

[4] "On the Act of Blinking," E. Ponder and W. P. Kennedy, Quarterly J. of Exper. Physiol. **18**, 89 (1927–1928).

[5] "The Eyelid Reflex as a Criterion of Ocular Fatigue," M. Luckiesh and Frank K. Moss, J. Exper. Psych. (scheduled for publication July, 1937).

[6] "Size of Pupil as a Possible Index of Ocular Fatigue," M. Luckiesh and Frank K. Moss, Am. J. Ophthal. **16**, 393 (1933).

these and other reasons, that pupillary phenomena are adequate criteria for detecting deleterious visual effects. It will be noted from Table I that the pupil was dilated approximately five percent after an hour of continuous reading. This fact may be regarded as symptomatic of ocular fatigue. However, it is significant that the gradual dilation due to visual effort was approximately the same under the unfiltered and filtered illumination. Furthermore, the absolute size of the pupil, under the two conditions of illumination, remains practically constant. Thus it is concluded that the difference in infrared energy in the two cases is not an important factor when appraised by pupillary phenomena.

Convergence reserve

Since it has been shown that the tonicity of the external ocular muscles is altered by the performance of exacting tasks under unfavorable visual conditions,[7] such as low levels of illumination, it is conceivable that the absorption of invisible radiation by the eye-media might also augment visual effort expended in seeing. Obviously, such a criterion for appraising the effects of infrared radiation upon visual function is a rather indirect one. However, as one of a series of criteria, it may possess some significance. In the present research the average amplitude of convergence reserve was determined from a series of thirty consecutive trials taken immediately before and after the period of reading by means of an ophthalmic ergograph. It will be noted from Table I that convergence reserve decreased about the same under the two experimental conditions. The relatively small decrease in convergence reserve observed in this experiment is consistent with the results obtained in a previous and comprehensive research.[7] The latter data establish the fact that the decrease in convergence reserve is not large when the reading is performed under favorable conditions.

Demonstration of visual test

This test involves a determination of the time required to recognize a series of letters formed by small breaks in a pattern of parallel diagonal lines printed in dark gray upon a somewhat lighter gray background.[8] The purpose of the pattern of parallel diagonal lines was not to present a maze in which the test-object was hidden, but rather to present a condition in which fixation is difficult. When such a pattern is viewed, the normal eye-movements are greatly accelerated in an involuntary manner. In general, this technique has been found to be highly sensitive to various factors which influence visual function. In the present experiment, the test was performed before and after the hour of reading. No significant difference in the time required to perform the test was observed under the two qualities of illumination.

Rate of reading

It is conceivable that any deleterious visual factor would alter the normal rate of reading. *A priori*, if the absorption of infrared radiation in the eye-media resulted in irritation and distraction, a differential in the rate of reading under the direct and filtered light might be expected. The data of Table I do not even suggest such an effect.

Summary

It is concluded from these data that no significant differences in visual function result from reading for one hour under 100 footcandles of direct illumination from tungsten-filament lamps and under the same intensity from the same sources obtained by filtering the light through three inches of water. Hence it follows that the presence of four times as much infrared energy caused no effects measurable by the various criteria used. It also seems reasonably safe to assume that the presence of larger amounts of infrared energy in artificial light than in natural light of the same intensity is not a factor of importance in lighting practice. It should be emphasized that these conclusions are based upon experimental results obtained with subjects possessing normal vision and free from pathologic conditions.

The authors acknowledge their indebtedness to Mr. S. K. Guth for his assistance in the prosecution of this research.

[7] "Fatigue of Convergence Induced by Reading as a Function of Illumination Intensity," M. Luckiesh and Frank K. Moss, Am. J. Ophthal. **18**, 319 (1935).

[8] "A Demonstrational Test of Vision," M. Luckiesh and Frank K. Moss, Am. J. Psych. **45**, 135 (1933).

The Effect of Irradiation on the Transmission of Lithium Fluoride

EDWIN G. SCHNEIDER
Harvard University, Cambridge, Massachusetts
(Received November 20, 1936)

The transmission of lithium fluoride in the visible and ultraviolet may be considerably reduced by exposure to very intense ultraviolet radiation, electron bombardment or contact with a low pressure electrical discharge. Strong absorption bands with maxima at 5200, 3100, and 2500A and a gradually increasing absorption below 1800A appear after a few minutes exposure to electron bombardment or contact with the low pressure discharge. Since 75 hours exposure to the light of a hydrogen discharge tube caused only about a maximum decrease of 5 percent in transmission, lithium fluoride may be considered as satisfactory as fluorite for use in ultraviolet optical instruments. Since all of the absorption bands build up at about the same rate, the visible discoloration, faint yellow to deep red, may be used as an indication of the change in the ultraviolet. Lithium fluoride colored by irradiation may be useful as a filter in ultraviolet studies.

A FAINT visible discoloration throughout the body of a lithium fluoride window was observed after a prolonged exposure to extreme ultraviolet radiation. Since a similar increase in the absorption in the ultraviolet might put a serious limitation on the use of this substance in many optical instruments, an investigation of the effects of irradiation seemed advisable.

A plate of lithium fluoride placed about two centimeters from the end of the capillary of a low pressure hydrogen discharge tube carrying about 50 m.a. at 25,000 volts showed a barely perceptible darkening after 35 hours exposure and was about the color of a Wratten K filter at the end of 75 hours. The yellow color was most intense on the side of the crystal which had been nearest the light source, the other side being almost clear. A test of the transmission of this colored specimen showed a decrease in transparency of less than five percent in the region below 1200A and broad absorption bands which decreased the transmission by about five percent at 2500, 3100 and 5200A. The two near ultraviolet bands have been observed in specimens of lithium fluoride exposed to x-rays;[1] the band at 5200A did not appear.

A crystal exposed to an electron beam produced in a cold cathode discharge tube by an induction coil giving approximately 100,000 volts developed all of these absorption bands. Since the region of discoloration was confined to a surface layer so thin that a few strokes on the polishing lap were sufficient to remove all traces of color, the depth of this layer was probably no greater than the range of the electrons in the crystal. The action of the electron beam was much more rapid than that of the ultraviolet light, a noticeable darkening appearing within a few seconds. The specimen became a deep red color by transmitted light and a bright blue by reflected light after a few minutes exposure.

An effect similar in all respects to that produced by electron bombardment occurred when crystals were placed in a discharge tube operated with hydrogen or air at pressures somewhat below 0.001 mm of mercury. Under these conditions the whole tube became filled with a blue glow. The coloring of the lithium fluoride by the discharge was probably due to the presence of a large number of relatively high speed electrons.

In order to eliminate the possibility that the color was due to material deposited by the discharge, a piece of glass was substituted for the lithium fluoride. Exposure for five hours produced no visible change.

Curves A and A′ show the transmission of two pieces of lithium fluoride after treatment with the glow discharge. The more transparent specimen was bombarded for five minutes and became a light orange. The other crystal, exposed for ten minutes, became about the color of ruby glass but appeared slightly turbid. The red crystal was so opaque to visible radiation below 5500A that

[1] Ottmer, Zeits. f. Physik **46**, 798 (1928).

no light could be seen through it with a hand spectroscope pointed directly at the sun. Curves B and B′ were taken after the crystals had been allowed to stand on a shelf exposed to air and the light of the room for two months. Curves C and C′ were taken a little over a year later.

A weakening of the absorption bands similar to that observed in the irradiated crystals which were kept for over a year can be produced rapidly by heating the crystal. Prolonged heating at a temperature just below red heat removed all of the discoloration produced by the ultraviolet light and restored the ultraviolet transmission of the specimen to its original value. In the case of the crystals treated by electron bombardment or glow discharge, the absorption bands could be completely removed by heating only if the visible color was a light yellow or fainter. The very deeply colored specimens showed a tenfold increase in transmission in the near ultraviolet and considerably less change in the visible and far ultraviolet when heated to a dull red for half an hour. Heating for longer periods at this temperature increased the transmission only slightly. Raising the temperature to the fusing point failed to remove all of the visible color.

A series of plates taken during the irradiation of the lithium fluoride by ultraviolet light and by glow discharge show that all of the absorption bands build up at about the same rate at first, the extension of the extreme ultraviolet cutoff increasing a little more slowly than the other bands. Prolonged irradiation by glow discharge increases the absorption in the visible to a greater extent than in the ultraviolet. Presumably ultraviolet irradiation would show the same effect if carried to the point where the crystal appeared red.

Before continuing the discussion, it might be advisable to state briefly a few of the observations of Gudden, Hilsch, Pohl and many others[2] on the behavior of the alkali halides. Irradiation of an alkali halide crystal with high energy electrons, x-rays or ultraviolet light of sufficiently short wave-length results in the formation of absorbing centers known as *F* centers. Only a limited concentration of *F* centers can be produced in a crystal. These absorbing centers may be bleached by the absorption of light or by heating, the bleaching process resulting in three changes of interest in the present discussion; (*a*) the reversion of the lattice to its original form, (*b*) the formation of colloidal particles of the alkali metal, and (*c*) the formation of *U* centers. The *U* centers have an absorption band lying just above the short wave-length cutoff of the normal crystal. Absorption of light by the *U* centers or heating a crystal containing *U* centers results in the formation of *F* centers. Heating a crystal containing colloidal metal may also result in the formation of *F* centers.

The *F* centers of lithium fluoride have an absorption maximum at 2500A.[1] Since the irradiation of the crystal was carried out with the full molecular spectrum of hydrogen, conditions were favorable for the repeated formation and bleaching by light of the *F* centers. Bleaching during the treatment with glow discharge or electron bombardment may have occurred through local heating by impact of the electrons. These processes should have resulted in the formation of colloidal lithium and *U* centers in the crystals.

The turbid appearance and the stability of the visible absorption band is consistent with the hypothesis that this band is due to colloidal lithium. Further evidence for this view is furnished by the fact that the visible absorbing centers may be produced in concentrations far beyond those of the *F* centers. The failure of the x-ray treatment to produce the visible absorption band may be attributed to the lack of radiation active in bleaching the *F* centers.

The absorption below 1800A produced by irradiation may be partially due to the formation of *U* centers. The gradual extension of the foot of the short wave-length limit rather than the development of a discrete band suggests that the greater part of this absorption is due to lattice distortion as the result of the formation of colloidal particles rather than to *U* centers. This is a reasonable assumption because lithium fluoride crystals containing small amounts of impurities show a similar extension of the ultraviolet ab-

[2] Since this material is widely scattered throughout the literature and many of the publications are not readily available, the reader is referred to a recent summary of this subject by A. L. Hughes, Rev. Mod. Phys. **8**, 294 (1936).

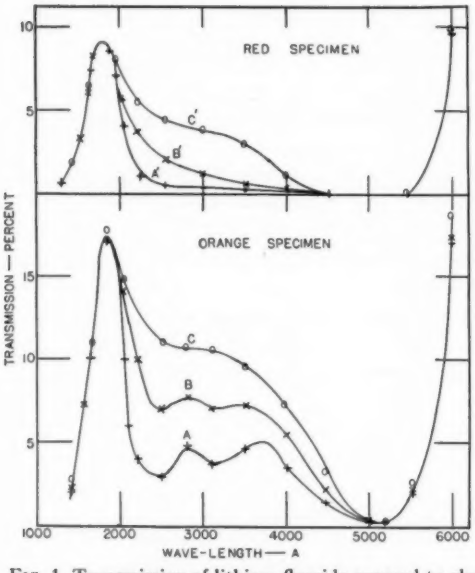


FIG. 1. Transmission of lithium fluoride exposed to glow discharge, *A*, *B* and *C* for 5 min., *A'*, *B'* and *C'* for 10 min. Curves *A* taken immediately after treatment, curves *B* after 2 months, curves *C* a year later.

sorption band. Since the shape of the extension is only slightly influenced by the nature of the impurity, the greater part of the effect may be ascribed to lattice distortion.

The cause of the band at 3100A is unknown.

In conclusion, it might be well to emphasize a few practical aspects of these experiments.

1. Since all of the absorption bands build up at about the same rate when a crystal of lithium fluoride is exposed to the general radiation from a discharge, the visible coloring may be used as an indication of the change in ultraviolet transmission.

2. Lithium fluoride windows should be placed so that they do not come in direct contact with the discharge.

3. The change in transmission of lithium fluoride on irradiation with ultraviolet light is negligible under ordinary conditions of use. Since fluorite turns purple at about the same rate that lithium fluoride becomes yellow, the latter may be used to replace fluorite in optical instruments.

4. Lithium fluoride colored by irradiation may be useful as a filter transparent in the ultraviolet but opaque in the visible. The low transmission in the ultraviolet will, however, limit the use of the colored salt for this purpose.

Corrections to "Changes in Color Temperature of Tungsten-Filament Lamps at Constant Voltage"

(J. O. S. A. **26**, 409 (1936))

DEANE B. JUDD,

National Bureau of Standards, Washington D. C.

IN this paper which appeared in the November (1936) issue, the following corrections should be made:

Eq. (7) should read:

$$R=(L/d_0^2)[\rho'(1-f)+\rho''(f/c^2)]. \tag{7}$$

Eq. (11) should read:

$$c^{3.1}=\rho' f/[\rho_0/0.99-\rho'(1-f)]. \tag{11}$$

In the text between these two equations wherever the terms I, θ, and ρ appear, the terms I'', θ'', and ρ'', respectively, should be substituted for them.

Hamilton's Method in Geometrical Optics

J. L. SYNGE
Department of Applied Mathematics, University of Toronto, Toronto, Canada
(Received October 29, 1936)

1. INTRODUCTION

SOME recent statements by Herzberger[1] regarding Hamilton's method in geometrical optics call for comment because they indicate a misconception which appears still to be fairly general. Broadly speaking, this misconception lies in the view that Hamilton invented a very elegant but inapplicable method in geometrical optics (the method of the characteristic function) which was replaced some sixty years later by the method of Bruns (the method of the eikonal) which, while resembling the method of Hamilton, differed from it in this respect, that the method of Bruns led to practical applications, whereas that of Hamilton did not.

In brief, the truth is that Bruns invented no new method, but rediscovered the method of Hamilton in a form less general, but sufficiently general for most practical applications; one method is as applicable as the other, because they are essentially the same. But it is also undoubtedly true that the modern development of the method dates from the work of Bruns, and that raises the interesting question: Why did seed sown by Hamilton prove sterile, but fertile when resown some sixty years later by Bruns?

The answer is, in part, that the seed of Hamilton fell on barren soil. No one eagerly investigated the practical possibilities of his work. His dynamical theory might have fallen into the same oblivion as his optical theory had not Jacobi become interested in the former and directed general attention to it, at the same time increasing its applicability by a slight remodeling.

But that is not the whole answer. Hamilton was himself partly to blame for the long oblivion to which his method was doomed. Swayed by the influence of Lagrange, whom he greatly admired, he sought to reduce the processes of geometrical optics entirely to analysis; hence no diagrams are to be found in his papers. Further, he had towards science the philosophical attitude of the eighteenth century rather than the materialistic attitude of many of his contemporaries. He sought to impress on geometrical optics the unifying principle of the characteristic function, much more for the beauty of the resulting theory than for the sake of facilitating practical calculations. That, at least, is the spirit apparent in almost every page of his published work, and it was not a spirit likely to arouse enthusiasm for his method in an age when practical considerations were becoming paramount.

The key to the practical application of his method is to be found in the papers published in his lifetime, but many who have looked with distaste through the endless pages of algebraic formulae may well have missed it altogether, so little prominence is given to it. To conclude that Hamilton had little interest in the practical applications would however be entirely false. Hamilton in private was very different from Hamilton in public, and his manuscripts reveal the enormous energy which he expended on the more practical aspects of his theory.

It is only about six years since a selection from his manuscripts has been published.[2] The use of his method by Hamilton in 1843 for the calculation of an achromatic double lens free from spherical aberration and coma[3] must surely be a convincing refutation of the charge that his method was inapplicable. We find also a detailed study of spherical aberration, coma and astigmatism in a general instrument of revolution,[4] as well as other details, some of which will be referred to later. Had Hamilton only brought himself to publish such a selection from his manuscripts during his lifetime, the history of geometrical optics during the past century and the world's judgment of the value of Hamilton's contribution to the subject might have been very different.

[1] M. Herzberger, J. Opt. Soc. Am. **26**, 177–178 (1936).

[2] *The Mathematical Papers of Sir W. R. Hamilton*, Vol. I (Cambridge, 1931), edited by A. W. Conway and J. L. Synge. This volume will be referred to as *H.*

[3] *H.*, 383–386.

[4] *H.*, 376–382.

Since long-established misconceptions die hard, it will be well to make some further remarks of a general nature before proceeding to the detail of Hamilton's method.

In 1832[5] Hamilton defined the characteristic function V of an optical system as the optical length of the ray joining two points, expressed as a function of their six coordinates. Excellent as V was from the point of view of general theory, its calculation for a given instrument presented insuperable difficulties, and Herzberger is right when he states that ". . . no one has succeeded until now, except for the case of a plane mirror, in finding this Hamiltonian function for a given optical system. . . ." But he proceeds to state that ". . . only a few scientists, for instance, Maxwell, Rayleigh, Clausius, and Larmor, have developed optical problems by Hamilton's method." If by "Hamilton's method" he means "direct use of the V function," he is quite right. But it is the purpose of the present note to destroy the erroneous impression that the V function was the only arrow in Hamilton's quiver.

Hamilton defined *four* functions, V, W, T and S. He reserved the name "characteristic function" for V, referring to W and T as "auxiliary functions." There is, however, no reason why we should not refer to them all as *characteristic functions*, since each of them *characterizes* the optical system, in the sense that two optical systems of different constructions but identical characteristic functions behave in precisely the same way.

Although Hamilton's definitions are more general, the functions V, W and T of Hamilton (1832) are, respectively, the point-eikonal, mixed-eikonal and angle-eikonal of Bruns (1895).[6] It is therefore difficult to see what Herzberger (reference 1, p. 178) means when he states that "he [Bruns] differed from Hamilton, and his method opened a fertile field for future investigation." Hamilton's T function was made to depend on only four variables[7] (if we omit the color index), when the initial and final media were homogeneous.

The invention of the word "eikonal," due to Bruns, has been a misfortune to geometrical optics, because the new *name* implied the existence of a new *method* where there was nothing essentially new. It has helped to deflect attention still further from the more penetrating and general ideas of Hamilton. Moreover, the derivation of the word (eikon = image) perpetuates the erroneous concept that geometrical optics is coextensive with the theory of the formation of images.

As Hamilton himself pointed out,[8] geometrical optics may be divided into two parts, the first dealing with the general properties of an optical system, a characteristic function being supposed known, and the second dealing with the calculation of a characteristic function for a definite system. No one has questioned the magnitude of Hamilton's contribution to the former of these domains. Indeed many have probably regretted the completeness of his work, general formula following general formula with apparently no hope of their application. It is easy to overlook a short communication[9] which is of more importance for the theory of the instrument of revolution than many pages of the more general developments. As regards the calculation of a characteristic function for a definite system, the best advertisement for a proposed method is its application in a specific case. This Hamilton hardly gave at all in the papers published in his lifetime, an omission which has done more than

[5] *H.*, 168.

[6] Cf. Czapski-Eppenstein, *Grundzüge der Theorie der optischen Instrumente* (Leipzig, 1924), 229, or H. Bruns, *K. sächs. Ges. d. Wiss. Abhand. math.-phys. Cl.*, **21**, 323–436 (1895). Bruns based his theory on the theorem of Malus, apparently not knowing that the method of the characteristic function applies in the case of crystalline refraction where the theorem of Malus no longer holds. His complete ignorance of Hamilton's optical work is extraordinary, especially in view of the following statement (p. 329): "Eine ganz ähnliche Rolle, wie der Hamilton'sche Ansatz in der Mechanik, spielt nun der Eikonalbegriff auf dem allerdings weit engeren Gebiete der geometrischen Optik; . . . ." This unification of geometrical optics and dynamics had of course been achieved long before by Hamilton, but in his hands the theory of geometrical optics was in some ways the more general of the two theories.

We have here an amusing instance of science groping in the dark: (1) Hamilton invents a general method in optics; (2) Hamilton applies this method to dynamics; (3) due to the work of Jacobi, Hamilton's dynamical method becomes well known; (4) Bruns, acquainted with Hamilton's dynamical method, applies it to optics, but only partially, because Hamilton's method for crystalline media never passed from stage (1) to stage (2), and hence was not suggested to Bruns by his knowledge of (2).

[7] *H.*, 182, 299.

[8] *H.*, 297.

[9] *H.*, 297–303.

anything else to lay his method open to the charge of inapplicability.

In preparing for modern use the original work of a great pioneer like Hamilton, departures from his terminology and notation should not be made *without reason*. On the other hand, it would be absurd to stick to archaic usages where they are clumsy. Thus we may, by vector or other notation, abbreviate long expressions with advantage, but to change Hamilton's V, for example, to another letter is to risk confusion for the sake of stamping one's signature in quite a trivial way on another man's theory.

2. General Theory for Any Media[10]

Two compact notations present themselves, the common vector notation and indicial notation, with the summation convention for repeated suffixes. The latter is to be preferred, because it possesses both the compactness of vectors and the advantages of an explicit notation.

Latin suffixes will be employed for the range 1, 2, 3, and Greek suffixes for the range 1, 2. Summation is implied when a suffix is repeated in a product: thus $\sigma_r\sigma_r=\sigma_1^2+\sigma_2^2+\sigma_3^2$, $\sigma_\rho\sigma_\rho=\sigma_1^2+\sigma_2^2$.

Each medium possesses a *medium function* v, a function of position, direction and frequency of the light under consideration; it is the *reciprocal of the ray-velocity*, or proportional thereto. Let us, for simplicity, suppress the dependence on frequency by dealing with monochromatic light throughout. Then

$$v=v(\alpha_r, x_r), \tag{2.1}$$

where x_r are rectangular Cartesian coordinates and α_r direction cosines, so that identically

$$\alpha_r\alpha_r=1. \tag{2.2}$$

By virtue of this relation, v may be expressed as a homogeneous function of α_r of degree unity: we shall suppose this done, so that

$$\alpha_r\partial v/\partial\alpha_r=v. \tag{2.3}$$

If v is independent of direction, the medium is *isotropic:* then

$$v=n(\alpha_r\alpha_r)^{\frac{1}{2}}, \tag{2.4}$$

where n is the index of refraction. If v is independent of position, the medium is *homogeneous*.

We define

$$\sigma_r=\partial v/\partial\alpha_r. \tag{2.5}$$

Elimination of α_r from these equations and (2.2) gives a relation of the form

$$\Omega(\sigma_r, x_r)=0. \tag{2.6}$$

We shall suppose that the function Ω is so expressed, by use of this relation, that $\Omega+1$ is homogeneous of degree unity in σ_r, so that identically

$$\sigma_r\partial\Omega/\partial\sigma_r=\Omega+1. \tag{2.7}$$

If the medium is isotropic, (2.6) reads

$$\Omega\equiv(\sigma_r\sigma_r)^{\frac{1}{2}}/n-1=0, \tag{2.8}$$

and if it is homogeneous x_r do not occur in Ω.

From (2.5) and (2.7) it follows that

$$\alpha_r/v=\partial\Omega/\partial\sigma_r. \tag{2.9}$$

The optical length of any curve is $\int v ds$ taken along it, ds being the element of arc. By Fermat's principle, a ray satisfies the stationary condition

$$\delta\int v ds=0, \tag{2.10}$$

and therefore a ray satisfies the differential equations

$$d\sigma_r/ds=\partial v/\partial x_r. \tag{2.11}$$

Fermat's principle also gives for the increments in σ_r at a surface of discontinuity (refraction or reflection)

$$\Delta\sigma_r=\theta l_r, \tag{2.12}$$

where θ is a factor of proportionality and l_r the direction cosines of the normal to the surface.

If light is regarded as a wave motion, propagated in accordance with the construction of Huygens, it is found that σ_r are proportional to the direction cosines of the normal to the wave element for which α_r are the direction cosines of the corresponding ray. If a vector is laid off along the normal to the wave surface, with a magnitude equal to the reciprocal of the normal wave velocity, then σ_r are precisely the components of this vector. For this reason Hamilton called them *components of normal slowness:* we shall refer to them briefly as *components*.

An *optical system* consists of an *initial medium* M' and a *final medium* M, separated by any

[10] As a general reference, see *H.*, 168 *et seq.*

number of media. Quantities pertaining to M' are marked with an accent and those pertaining to M are left unaccented.[11] Thus we have for M relations as written above, and for M' the same relations accented. The axes of x_r' need not be the same as those of x_r.

The *V function* of the system is defined as the optical length of the ray passing from a point x_r' of M' to a point x_r of M, considered as a function of these six coordinates, so that

$$V = V(x_r', x_r). \tag{2.13}$$

Arbitrary variation of the end points gives

$$\left.\begin{aligned} &\delta V = -\sigma_r'\delta x_r' + \sigma_r \delta x_r, \\ &\sigma_r' = -\partial V/\partial x_r', \qquad \sigma_r = \partial V/\partial x_r. \end{aligned}\right\} \tag{2.14}$$

Consequently V satisfies the two partial differential equations

$$\Omega'(-\partial V/\partial x_r', x_r') = 0, \quad \Omega(\partial V/\partial x_r, x_r) = 0. \tag{2.15}$$

If the V function of the system is given, then we can find the directions at x_r', x_r of the ray joining them: for (2.14) give σ_r', σ_r and (2.9) and its accented companion give α_r, α_r'.

The *W function* is defined by[12]

$$W = V - \sigma_r x_r. \tag{2.16}$$

Here W is expressed as a function of the nine quantities x_r', x_r, σ_r, but passing to a neighboring ray we have by (2.14)

$$\delta W = -\sigma_r'\delta x_r' - x_r \delta\sigma_r, \tag{2.17}$$

which shows that W is expressible in the form

$$W = W(x_r', \sigma_r). \tag{2.18}$$

Since $\delta x_r'$ are arbitrary in (2.17), we have

$$\sigma_r' = -\partial W/\partial x_r', \tag{2.19}$$

and hence W satisfies the partial differential equation

$$\Omega'(-\partial W/\partial x_r', x_r') = 0. \tag{2.20}$$

The *T function* of the system is defined by[13]

$$T = V + \sigma_r' x_r' - \sigma_r x_r. \tag{2.21}$$

[11] It is a nicety of Hamilton's notation that the accent refers to the *initial* medium; this economizes the use of accents, since most of our discussion is likely to be concerned with the final medium.

[12] The sign of Hamilton's W (*H.*, 175) is here reversed to simplify physical interpretation.

[13] The sign of Hamilton's T (*H.*, 175) is here reversed to simplify physical interpretation.

Passing to a neighboring ray, we have

$$\delta T = x_r'\delta\sigma_r' - x_r\delta\sigma_r, \tag{2.22}$$

which shows that T is expressible in the form

$$T = T(\sigma_r', \sigma_r). \tag{2.23}$$

3. Heterogeneous Media

Let M' and M be heterogeneous, so that v' and Ω' involve x_r', and v and Ω involve x_r. It is then possible to pass from a given ray to a neighbouring ray with arbitrary variations $\delta x_r'$, $\delta\sigma_r$. Hence the differentials in (2.17) are independent, and we have

$$\sigma_r' = -\partial W/\partial x_r', \qquad x_r = -\partial W/\partial\sigma_r. \tag{3.1}$$

If the W function of the system is known, we can thus obtain the values of σ_r', x_r corresponding to arbitrary assigned values of x_r', σ_r. W satisfies the two partial differential equations

$$\Omega'(-\partial W/\partial x_r', x_r') = 0, \ \Omega(\sigma_r, -\partial W/\partial\sigma_r) = 0. \tag{3.2}$$

Also it is possible to pass from a given ray to a neighboring ray with arbitrary variations $\delta\sigma_r'$, $\delta\sigma_r$. Hence the differentials in (2.22) are independent, and we have

$$x_r' = \partial T/\partial\sigma_r', \qquad x_r = -\partial T/\partial\sigma_r. \tag{3.3}$$

If the T function of the system is known, we can thus obtain the values of x_r', x_r corresponding to arbitrary values of σ_r', σ_r. T satisfies the two partial differential equations

$$\Omega'(\sigma_r', \partial T/\partial\sigma_r') = 0, \quad \Omega(\sigma_r, -\partial T/\partial\sigma_r) = 0. \tag{3.4}$$

4. Homogeneous Media

Let M' and M be homogeneous, but not necessarily isotropic. There is no restriction at all on the nature of the media intervening between M' and M.

We shall here recapitulate the formulae of §2 for M' and M, introducing the simplifications consequent on homogeneity. We have

$$\alpha_r'\alpha_r' = 1, \qquad \alpha_r\alpha_r = 1; \tag{4.1}$$

$$\left.\begin{aligned} v' = v'(\alpha_r'), \qquad & v = v(\alpha_r), \\ \alpha_r'\partial v'/\partial\alpha_r' = v', \qquad & \alpha_r\partial v/\partial\alpha_r = v; \end{aligned}\right\} \tag{4.2}$$

$$\sigma_r' = \partial v'/\partial\alpha_r', \qquad \sigma_r = \partial v/\partial\alpha_r; \tag{4.3}$$

$$\Omega'(\sigma_r') = 0, \qquad \Omega(\sigma_r) = 0; \tag{4.4}$$

$$\sigma_r'\partial\Omega'/\partial\sigma_r'=1, \qquad \sigma_r\partial\Omega/\partial\sigma_r=1; \tag{4.5}$$

$$\alpha_r'/v'=\partial\Omega'/\partial\sigma_r', \qquad \alpha_r/v=\partial\Omega/\partial\sigma_r. \tag{4.6}$$

By (2.11), $\sigma_r'=const.$, $\sigma_r=const.$ along the initial and final portions of a ray, which are straight. Also

$$\sigma_r'=-\partial V/\partial x_r', \qquad \sigma_r=\partial V/\partial x_r, \tag{4.7}$$

and V satisfies the two partial differential equations

$$\Omega'(-\partial V/\partial x_r')=0, \qquad \Omega(\partial V/\partial x_r)=0. \tag{4.8}$$

The components σ_r' are not independent, on account of (4.4), and σ_r are not independent. We can solve (4.4) to give

$$\sigma_3'=\sigma_3'(\sigma_\rho'), \qquad \sigma_3=\sigma_3(\sigma_\rho), \tag{4.9}$$

remembering that Greek suffixes have the range 1, 2.

By (4.9) W and T may be expressed in the forms

$$W=W(x_r', \sigma_\rho), \qquad T=T(\sigma_\rho', \sigma_\rho). \tag{4.10}$$

We shall in future suppose them so expressed.

Now in (2.17) the variations $\delta x_r'$, $\delta\sigma_\rho$ are independent, $\delta\sigma_3$ being given by (4.9). Hence

$$\begin{aligned} \sigma_r'&=-\partial W/\partial x_r', \\ x_\rho+x_3\partial\sigma_3/\partial\sigma_\rho&=-\partial W/\partial\sigma_\rho. \end{aligned} \tag{4.11}$$

If the W function of the system is known, and x_r', σ_ρ are assigned for a ray, the first of (4.11) gives us σ_r' and the second gives the two equations of the final portion of the ray, connecting x_r.

The W function satisfies the single partial differential equation

$$\Omega'(-\partial W/\partial x_r')=0. \tag{4.12}$$

In general[14] the variations $\delta\sigma_\rho'$, $\delta\sigma_\rho$ in (2.22) are independent, $\delta\sigma_3'$, $\delta\sigma_3$ being given in terms of them by (4.9). Hence

$$\begin{aligned} x_\rho'+x_3'\partial\sigma_3'/\partial\sigma_\rho'&=\partial T/\partial\sigma_\rho', \\ x_\rho+x_3\partial\sigma_3/\partial\sigma_\rho&=-\partial T/\partial\sigma_\rho. \end{aligned} \tag{4.13}$$

If the T function of the system is known, and the initial and final components are assigned, these are the equations of the initial and final portions of the ray.

T satisfies no partial differential equation at all, a fact wherein lies the importance of this function.

The S *function* is defined as[15]

$$S=T-\sigma_3'x_3'+\sigma_3x_3. \tag{4.14}$$

Then

$$\delta S=-\sigma_3'\delta x_3'+x_\rho'\delta\sigma_\rho'+\sigma_3\delta x_3-x_\rho\delta\sigma_\rho. \tag{4.15}$$

S is therefore expressible in the form

$$S=S(x_3', \sigma_\rho', x_3, \sigma_\rho). \tag{4.16}$$

In general[14] the variations $\delta x_3'$, $\delta\sigma_\rho'$, δx_3, $\delta\sigma_\rho$ are independent, and hence

$$\left.\begin{aligned} \sigma_3'&=-\partial S/\partial x_3', & \sigma_3&=\partial S/\partial x_3, \\ x_\rho'&=\partial S/\partial\sigma_\rho', & x_\rho&=-\partial S/\partial\sigma_\rho. \end{aligned}\right\} \tag{4.17}$$

If the S function of the system is known, and the components of the initial and final portions of a ray are assigned, together with planes assigned by given values of x_3', x_3, then the second line in (4.17) gives the points at which the ray cuts these planes.

The physical interpretations of W, T, S are as follows. $W(x_r', \sigma_\rho)$ is the optical length of a ray from the point x_r' to the intersection of the ray with the plane $\sigma_r x_r=0$. $T(\sigma_\rho', \sigma_\rho)$ is the optical length of a ray between its intersections with the planes $\sigma_r'x_r'=0$, $\sigma_r x_r=0$. $S(x_3', \sigma_\rho', x_3, \sigma_\rho)$ is the optical length of a ray between its intersections with planes drawn through $(0, 0, x_3')$, $(0, 0, x_3)$ with normals having direction ratios σ_r', σ_r, respectively.

5. Homogeneous Isotropic Media

Let M' and M be homogeneous and isotropic, without restriction on the nature of the intervening media. All the results of §4 hold here, with some simplifications. Denoting the refractive indices of M', M by n', n, respectively, and confining our attention to the general case (see reference 14), the basic formulae are as follows:

$$\alpha_r'\alpha_r'=1, \qquad \alpha_r\alpha_r=1; \tag{5.1}$$

[14] The most noteworthy exception is the prismatic combination, in which all the media involved are homogeneous and the surfaces of separation are planes; cf. *H.*, 177. Another exception is that in which the system consists of two media separated by a developable surface. The test of independence is this: does a congruence of parallel rays incident on the system emerge as a congruence whose directions require two parameters for their description? If so, we have the required independence, for then a ray exists having arbitrary values of σ_ρ', σ_ρ.

[15] *H.*, 268. The sign of Hamilton's S has been reversed, to accord with the reversals already made in the signs of W and T.

$$\sigma_r' = n'\alpha_r', \qquad \sigma_r = n\alpha_r; \tag{5.2}$$

$$\sigma_3'^2 = n'^2 - \sigma_\rho'\sigma_\rho', \qquad \sigma_3^2 = n^2 - \sigma_\rho\sigma_\rho; \tag{5.3}$$

$$\left.\begin{aligned} V &= V(x_r', x_r), \\ \sigma_r' &= -\partial V/\partial x_r', \qquad \sigma_r = \partial V/\partial x_r; \end{aligned}\right\} \tag{5.4}$$

$$\frac{\partial V}{\partial x_r'}\frac{\partial V}{\partial x_r'} = n'^2, \qquad \frac{\partial V}{\partial x_r}\frac{\partial V}{\partial x_r} = n^2; \tag{5.5}$$

$$\left.\begin{aligned} W &= W(x_r', \sigma_\rho), \\ \sigma_r' &= -\partial W/\partial x_r', \\ x_\rho - x_3\sigma_\rho/\sigma_3 &= -\partial W/\partial \sigma_\rho; \end{aligned}\right\} \tag{5.6}$$

$$\frac{\partial W}{\partial x_r'}\frac{\partial W}{\partial x_r'} = n'^2; \tag{5.7}$$

$$\left.\begin{aligned} T &= T(\sigma_\rho', \sigma_\rho), \\ x_\rho' - x_3'\sigma_\rho'/\sigma_3' &= \partial T/\partial \sigma_\rho', \\ x_\rho - x_3\sigma_\rho/\sigma_3 &= -\partial T/\partial \sigma_\rho; \end{aligned}\right\} \tag{5.8}$$

$$\left.\begin{aligned} S &= S(x_3', \sigma_\rho', x_3, \sigma_\rho), \\ x_\rho' &= \partial S/\partial \sigma_\rho', \qquad x_\rho = -\partial S/\partial \sigma_\rho. \end{aligned}\right\} \tag{5.9}$$

A minor point requiring care is the ambiguous sign introduced on taking a square root in (5.3): the sign must be fixed by direct consideration of the system.

The physical interpretations of W, T, S are as follows. $W(x_r', \sigma_\rho)$ is the optical length of a ray from the point x_r' to the foot of the perpendicular dropped from the origin O in M on the final portion of the ray. $T(\sigma_\rho', \sigma_\rho)$ is the optical length of a ray between the feet of the perpendiculars dropped on it from the origins O', O in M', M, respectively. $S(x_3', \sigma_\rho', x_3, \sigma_\rho)$ is the optical length of ray between the feet of the perpendiculars dropped on it from the points $(0, 0, x_3')$, $(0, 0, x_3)$.

6. Calculation of the T Function for a Pair of Homogeneous Media[16]

Let the system consist of homogeneous media M', M (not necessarily isotropic), separated by a surface Σ, which is not developable. We shall here, and henceforward, use a single set of coordinate axes in M' and M.

Let the equation of Σ be put in the form

$$x_3 = f(x_\rho). \tag{6.1}$$

Let Δ denote an increment on crossing Σ, so that

$$\Delta\sigma_r = \sigma_r - \sigma_r'. \tag{6.2}$$

Then by (2.12)

$$\Delta\sigma_\rho/\Delta\sigma_3 = -\partial f/\partial x_\rho. \tag{6.3}$$

These equations may be solved to give x_ρ (and consequently by (6.1) x_3) as functions of $\Delta\sigma_r/\Delta\sigma_3$. Thus

$$x_\rho = g_\rho(\Delta\sigma_r/\Delta\sigma_3), \qquad x_3 = f[g_\rho(\Delta\sigma_r/\Delta\sigma_3)]. \tag{6.4}$$

Since T is a function of σ_ρ', σ_ρ and these are constant on the initial and final portions of the ray, the value of T, as given by (2.21), is independent of the choice of the points x_r', x_r on these portions. Let us take these points coincident on Σ. Then $V = 0$, and

$$T = (\sigma_r' - \sigma_r)x_r = -x_r\Delta\sigma_r. \tag{6.5}$$

Hence by (6.4), *the T function for a pair of homogeneous media is*

$$T = -\Delta\sigma_\rho \cdot g_\rho(\Delta\sigma_r/\Delta\sigma_3) - \Delta\sigma_3 f[g_\mu(\Delta\sigma_r/\Delta\sigma_3)]. \tag{6.6}$$

Here $\Delta\sigma_3$ is to be expressed in terms of σ_ρ', σ_ρ by means of (4.4) for anisotropic media or (5.3) for isotropic media.

The following method is an alternative. Let the tangential equation of Σ be put in the form

$$p = \phi(l_\rho/l_3), \tag{6.7}$$

where p is the perpendicular from the origin on the tangent plane to Σ which has a normal with direction cosines l_r. Then by (6.5), (2.12)

$$\left.\begin{aligned} T &= -\theta x_r l_r = \pm p\theta, \\ \Delta\sigma_r\Delta\sigma_r &= \theta^2, \qquad \Delta\sigma_\rho/\Delta\sigma_3 = l_\rho/l_3, \end{aligned}\right\} \tag{6.8}$$

and so

$$T = \pm(\Delta\sigma_r\Delta\sigma_r)^{\frac{1}{2}}\phi(\Delta\sigma_\rho/\Delta\sigma_3). \tag{6.9}$$

The ambiguous sign must be fixed by direct consideration of the system.

For example, if the surface Σ is a paraboloid of revolution

$$x_3 = \tfrac{1}{2}x_\rho x_\rho/R, \tag{6.10}$$

(6.6) gives[17]

$$T = \tfrac{1}{2}R\Delta\sigma_\rho\Delta\sigma_\rho/\Delta\sigma_3, \tag{6.11}$$

[16] *H.*, 214. The method given by T. Smith (Trans. Opt. Soc. **28**, 264 (1926–27)) is a variant of Hamilton's method based on Eq. (6.7) of the present paper. Smith was apparently unaware that the problem had been solved (and the solution published) nearly a century earlier by Hamilton, nor did he realize that he was actually finding the tangential equation of a surface by the use of homogeneous coordinates. The examples given by Smith can be worked out with as great, or greater, ease by direct application of Hamilton's method.

[17] *H.*, 216.

and if $\sum$ is a sphere

$$x_r x_r = R^2, \tag{6.12}$$

(6.9) gives[18]

$$T = \pm R(\Delta\sigma_r \Delta\sigma_r)^{\frac{1}{2}}. \tag{6.13}$$

The formulae (6.11), (6.13) hold whether the media are anisotropic or isotropic, and whether refraction or reflection occurs. It is only when we come to substitute for $\Delta\sigma_3$ in terms of $\sigma_\rho'\sigma_\rho$ that distinctions must be made.

7. Calculation of the T Function for any System of Homogeneous Media[19]

Let us suspend the convention as to range and summation for suffixes enclosed in brackets.

Let there be a system of homogeneous media $M^{(0)}$, $M^{(1)}$, $\cdots M^{(n)}$, separated by surfaces of discontinuity (not all plane). We shall employ one set of axes for all the media. Each pair of media has a T function, that for $M^{(i)}$, $M^{(i+1)}$ being $T^{(i,\,i+1)}$, to be found as in §6. The T-function for the complete system will then be

$$T = \sum_{i=0}^{n-1} T^{(i,\,i+1)}. \tag{7.1}$$

Since $T^{(i,\,i+1)}$ is a function of $\sigma_\rho^{(i)}$, $\sigma_\rho^{(i+1)}$, the components in $M^{(i)}$, $M^{(i+1)}$, respectively, the expression (7.1) involves $\sigma_\rho^{(i)}$ $(i=0,\ 1,\ \cdots n)$. The components in the intermediate media are to be eliminated by means of the *stationary conditions*

$$\partial T/\partial\sigma_\rho^{(i)} = 0 \qquad (i=1,\ 2,\ \cdots n-1). \tag{7.2}$$

When these $2n-2$ equations are solved for the $2n-2$ quantities $\sigma_\rho^{(i)}$ $(i=1,\ 2,\ \cdots n-1)$, and the results substituted in (7.1), we have the T function for the complete system in the form

$$T = T(\sigma_\rho^{(0)},\ \sigma_\rho^{(n)}), \tag{7.3}$$

a function of initial and final components.

The elimination required above is likely to prove difficult, if an exact form for T is required.

[18] *H.*, 389.
[19] *H.*, 217, 408 *et seq.*, 492.

8. Approximate Theory for the Instrument of Revolution

Let there be an instrument of revolution, the surfaces of discontinuity being of revolution about the x_3 axis. The media are homogeneous, but not necessarily isotropic; however the anisotropy must conform to the symmetry of revolution, which means that v in each medium must depend only on the inclination of the ray to the axis of symmetry. We shall use a single set of axes.

Putting

$$\epsilon' = \sigma_\rho'\sigma_\rho', \qquad \epsilon_{,} = \sigma_\rho'\sigma_\rho, \qquad \epsilon = \sigma_\rho\sigma_\rho, \tag{8.1}$$

it follows from symmetry that T is a function of ϵ', $\epsilon_{,}$, ϵ. Expanding in a power series and retaining terms of the fourth order, but neglecting higher powers (the values of σ_ρ', σ_ρ being supposed small), we have[20]

$$\left.\begin{aligned} &T = T_{(0)} + T_{(2)} + T_{(4)}, \\ &T_{(0)} = \text{const.}, \\ &T_{(2)} = P'\epsilon' + P_{,}\epsilon_{,} + P\epsilon, \\ &T_{(4)} = Q''\epsilon'^2 + Q_{,,}\epsilon_{,}^2 + Q\epsilon^2 + Q_{,}'\epsilon'\epsilon_{,} \\ &\qquad\qquad + Q'\epsilon'\epsilon + Q_{,}\epsilon_{,}\epsilon, \end{aligned}\right\} \tag{8.2}$$

the P's and Q's being constants. The gaussian optics of the instrument depends on the P's and its aberrations on them and on the Q's.

If the initial and final media are *isotropic* (although the intermediate media are not necessarily so), we have the following values of x_3 for cardinal points,[21] if the initial and final portions of the ray proceed in the positive sense of x_3:

Focal points: $(-2P'n',\ 2Pn)$;
Nodal points: $(-2P'n' - P_{,}n,\ 2Pn + PP_{,}n')$
Principal points: $(-2P'n' - P_{,}n',\ 2n + P_{,})n$;.

The conditions[22] for the formation of a point image (quantities of the fifth order being neglected) of any object at infinity are

$$P = 4Qn^2, \qquad Q_{,} = 0, \qquad Q_{,,} = 0, \tag{8.3}$$

the defects corresponding to violation of these conditions being respectively spherical aberration, coma and astigmatism.

[20] *H.*, 298.
[21] *H.*, 395, 508. We must remember that the T of the present paper is the negative of that employed in *H.*
[22] *H.*, 378, 509.

Hamilton did not consider curvature of the image and distortion. The condition for a flat image[23] is

$$Q' = 0, \tag{8.4}$$

and the condition for no distortion[23] is

$$P_{,} = 2Q_{,}'n'^{2}. \tag{8.5}$$

If we wish to deal with aberrations corresponding to object points at finite distance, it is better to use the W function, which on account of symmetry will be a function of

$$x_\rho' x_\rho', \quad x_\rho' \sigma_\rho, \quad \sigma_\rho \sigma_\rho, \quad x_3',$$

subject to the partial differential equation (4.12) in the case of an anisotropic initial medium and (5.7) in the case of an isotropic one.

9. Calculation of the T Function up to the Fourth Order for an Instrument of Revolution

The calculation of the T function for an instrument of revolution according to the method of §7 is much simplified when we are satisfied with an accuracy of the fourth order in T. Supposing $T^{(i,\ i+1)}$ calculated to the fourth order for each pair of successive media $M^{(0)}$, $M^{(1)}$, $\cdots M^{(n)}$, we may put

$$T^{(i,\ i+1)} = T_{(0)}{}^{(i,\ i+1)} + T_{(2)}{}^{(i,\ i+1)} + T_{(4)}{}^{(i,\ i+1)}, \quad (i = 0, 1, \cdots n-1), \tag{9.1}$$

where the subscript indicates degree of homogeneity in the components $\sigma_\rho{}^{(i)}$, $\sigma_\rho{}^{(i+1)}$. The T function for the whole instrument may then be written

$$\begin{aligned} T &= T_{(0)} + T_{(2)} + T_{(4)}, \\ T_{(0)} &= \sum_{i=0}^{n-1} T_{(0)}{}^{(i,\ i+1)}, \\ T_{(2)} &= \sum_{i=0}^{n-1} T_{(2)}{}^{(i,\ i+1)}, \\ T_{(4)} &= \sum_{i=0}^{n-1} T_{(4)}{}^{(i,\ i+1)}. \end{aligned} \tag{9.2}$$

It would appear at first sight that in the equations (7.2), which have to be employed to eliminate the intermediate components, terms of the third order ought to be retained in order to secure accuracy of the fourth order in T. That is not the case:[24] it is sufficient to employ instead of (7.2) the equations

$$\partial T_{(2)} / \partial \sigma_\rho{}^{(i)} = 0, \qquad (i = 1, 2, \cdots n-1), \tag{9.3}$$

which are linear and homogeneous in $\sigma_\rho{}^{(i)}$ $(i = 0, 1, \cdots n)$.

The following calculations of T, accurate to the fourth order, were carried out by Hamilton:

General thin system of revolution involving n surfaces (*H.*, 455).
Single thin lens of isotropic material (*H.*, 375).
Single thin lens of uniaxial crystal (*H.*, 337).
Single thick lens (*H.*, 401).
Two thin lenses close together (*H.*, 373, 405).
Two thick lenses (*H.*, 402, 458).

Of these, only the third was published in Hamilton's lifetime.

For a general method of calculating T to the fourth order, following the lines indicated by Hamilton, see *H.*, 512.

[23] *H.*, 510.

[24] *H.*, 390, 408, 507.

Characteristics of Some New Mercury Arc Lamps

B. T. Barnes and W. E. Forsythe
Incandescent Lamp Department, General Electric Company, Nela Park, Cleveland, Ohio
(Received December 19, 1936)

UNTIL the last few years, the only mercury lamps in common use for lighting were of the low pressure type[1] developed by Cooper-Hewitt. Although these lamps have over-all luminous efficiencies (including wattage dissipated in auxiliary equipment) of the same order of magnitude as tungsten lamps of the same wattage, they have been used little except in industrial plants. There are several reasons for this fact: ambient air temperatures below 15°F cause starting difficulties and low luminous efficiency, making year-round outdoor use in this climate impractical; the almost complete absence of red in their spectrum makes them unsuitable for general illumination whenever the appearance of objects of various colors is of importance; and the size of the lamp and of the auxiliary equipment is such as to discourage certain applications. The susceptibility to cold surroundings may be decreased by the use of an enclosing fixture and the deficiency in red may be counteracted by using the lamps in combination with neon lamps or tungsten filament lamps. Nevertheless the Cooper-Hewitt type of lamp probably will find little demand in the future because there are now available mercury lamps of higher efficiency, better color, smaller size and decreased susceptibility to changes in ambient temperature.

Some of these newer mercury lamps have an arc length of 10 to 20 centimeters and an arc tube diameter of 2 to 4 centimeters. There is an oxide-coated or impregnated electrode[2] at either end of the arc tube. Such lamps are intended only for a.c. operation. They are supplied with a measured quantity of mercury which is entirely vaporized at the normal temperature of operation. This feature makes the pressure, and consequently the light output, nearly independent of the temperature and the rate of circulation of the surrounding air.

Table I gives data on type H-1 and type H-2 lamps, which are both of the sort described in the preceding paragraph. Since only four type H-1 and two type H-2 lamps were used, the values given in the table should not be assumed to be strictly representative of lamps of these types. Data for a Cooper-Hewitt lamp are also given. Since the latter and the type H-2 lamp have nearly the same wattage input, differences between the efficiencies of production of a spectral line by the two lamps are evident on inspection. The lines at wave-lengths 5461, 4358, 4047, and 10,140A are all stronger in the Cooper-Hewitt, while the 5770–91A group is stronger in the type H-2 lamp. If the electrons have a Maxwellian[3] distribution of velocities, this indi-

TABLE I. *Operating characteristics and radiation output of four mercury lamps.*

Lamp		Cooper Hewitt	Type H-1	Type H-2	Type H-3
Inside tube diameter (cm)		2.4	3.5	2.5	.4
Arc length (cm)		128	16.0	10.5	1.8
Vapor pressure (atm)		.0003	1.4	.4	28.5
Volts		71.9	150	67.5	251
Amp.		3.70	2.77	4.06	.401
Watts*		266	400	250	85.6
Lumens		6500	16000	7000	3400
Lumens per watt*		24.5	40	28	39.5
Candles per mm²†		.031	1.4	1.0	18
Principal Lines	Band of Wave-lengths	Spectral Intensities (μw per cm² at One Meter Distance from Lamp Axis)			
17100–16900			34.4	11.9	6.9
15300			27.1	7.9	6.1
13955–13570			44	14.8	8.5
12100–11900			27.4	8.2	6.3
11289			35.7	12.5	7.8
10140		34.3	69	22.7	10.8
	7420–6830			3.3	3.75
	6830–6350		5.7	2.3	3.0
	6350–5950		5.6	1.9	3.1
5791–5770	5950–5615	26.3	127	55	19.2
5461	5615–5320	81	131	57.5	33.5
	5320–5060		3.5	1.24	2.2
	5060–4840		3.02	1.47	2.05
	4840–4650			.63	2.2
	4650–4475			.62	2.6
4358		82	92	44.5	27.1
4047		59	44.5	23.6	14.5
3663–3650		30	41	22.5	25.7
3341		2.7			3.2
3129		15			6.9
3022		.28			1.21
2967		.10			.73

* Exclusive of losses in auxiliary equipment.
† Maximum value at center of arc.

[1] L. J. Buttolph, R.S.I. **1**, 487 (1930).
[2] J. W. Ryde, Elec. Rev. (London) **113**, 583 (1933); H. E. Krefft and E. Summerer, Das Licht **4**, 23 (1934).
[3] K. T. Compton and I. Langmuir, Rev. Mod. Phys. **2**, 204 (1930).

TABLE II. *Operating characteristics and percentage of input radiated in various ultraviolet lines for a bare capillary arc tube and a bare Uviarc.*

	Inside Tube Diameter (cm)	Arc Length (cm)	Vapor Pressure (atm.)	Volts	Amp.	Watts*
Bare capillary tube	.4	1.8	28.5	250	.403	85.6
d.c. Uviarc for 110v line	1.5	8.5	1.0	75	3.75	281

	Percent of Input Radiated in Spectral Lines	
Wave-length	Bare capillary tube	d.c. Uviarc for 110v line
3663–3650	2.85	3.66
3341	.40	.27
3132–3126	1.73	2.53
3027–3021	.76	1.12
2967	.68	.55
2925	.19	.068
2894	.30	.20
2806–2803	.31	.39
2753	.24	.123
2701–2697	.23	.164
2655–2652	.37	.83
2537–2535	.064	1.14
2484–2482	.14	.33
2464	.065	.052
2447	.031	.043
2401–2399	.063	.142
2380–2378	.079	.128
2353	.089	.076
2323	.085	.047
2302	.084	.076
2259	.039	.048
Erythemal U.V.†	1.92	2.82
Erythemal U.V. of λ's >2800A	1.38	1.37
Erythemal U.V. of λ's <2800A	.54	1.45

* For arc tube only.
† Equivalent of λ2967A for producing a "minimum perceptible erythema."

cates that the Cooper-Hewitt lamp has a lower mean electron velocity than the type H-2. The relatively low intensity of the 3650–63A group in the radiation from type H-2 lamps is not necessarily in contradiction to this conclusion because the radiation of this wave-length will suffer more absorption in the bulb of the type H-2 lamp on account of the much higher glass temperature.

Another type of mercury lamp developed recently[4] has a quartz capillary arc tube with a length of only a few centimeters and an internal diameter of a few millimeters. Solid electrodes are used but there is ordinarily a small amount of liquid mercury still present when the lamp is in operation. The capillary arc tube is mounted inside a bulb which shields it from air currents and absorbs most of the ultraviolet radiation of wave-lengths less than 3200A. These lamps are designed to operate at a mercury vapor pressure of several, in some cases as high as forty, atmospheres.

[4] Das Licht **5**, 84 (1935); C. Bol, Ingenieur **50**, 91 (1935).

The type H-3 lamp, for which data are given in Table I, is a capillary lamp of the sort described above. The values in the table are the averages for two lamps. The relative intensities of the principal spectral lines indicate that the mean electron velocity for this lamp is intermediate between the mean velocities for the H-1 and H-2 lamps and that for the Cooper-Hewitt lamp.

The energy radiated by the Cooper-Hewitt lamp is practically all line radiation. The H-1, H-2 and H-3 lamps all furnish continuous radiation of moderate intensity which is spread more or less uniformly throughout the visible spectrum. The continuous radiation becomes of more importance as the mercury pressure is increased. In the H-1 lamp about 8 percent of the light is in the continuous spectrum; in the H-3 about 25 percent.

The brightness of the arc also increases rapidly with pressure. Aside from increases in luminous efficiency, this is due to two fundamental characteristics of all arcs, namely, increasing gradient and decreasing arc diameter when the pressure is raised and the current is kept at the same order of magnitude. Thus, a high pressure arc will have a shorter tube than a low pressure arc of the same voltage and it may have a tube of smaller diameter if this does not make the glass temperature excessively high. The diameter of the arc tube could not be made much smaller in types H-1, H-2, or H-3 lamps without making the glass so hot that a decrease in the life or the lumen maintenance, or both, would result. On the other hand, the tube wall of the Cooper-Hewitt lamp becomes only moderately warm.

The discharge fills the Cooper-Hewitt tube fairly uniformly. Its brightness at the center of the arc is about the same as that of tungsten at 1740°K. The discharge in all the other lamps for which data are given in Table I concentrates near the axis of the tube. In the case of the type H-3 lamp, the brightness at the center of the tube is approximately the same as that of tungsten at 3150°K, nearly 600 times the brightness of the Cooper-Hewitt lamp. For certain technical uses of the lamp this high brightness is advantageous; for general use it must be counteracted by making the enclosing bulb diffusing or by

employing a diffusing or indirect illumination fixture.

In the type H-3 lamp most of the radiation of wave-lengths less than 3200A is absorbed by the outer bulb. By using a bulb of special ultraviolet transmitting glass, a very efficient source of erythemal ultraviolet could be obtained. This is indicated by the data in Table II obtained on a bare capillary arc tube. This tube was specially constructed to operate at 250 volts in open air—the tubes in type H-3 lamps would run at a lower voltage if the enclosing bulb were removed. Table II also gives, for the sake of comparison, the percentage of the input radiated in each ultraviolet line or group of lines by a commercial quartz mercury arc[1] operating at a pressure of about one atmosphere. It is evident that the capillary tube has about the same efficiency as the Uviarc for producing ultraviolet of wave-lengths greater than 2800A. if the radiation is weighted according to its effectiveness in producing a minimum perceptible erythema. The Uviarc is considerably more efficient in producing radiation of shorter wave-lengths.

Specially constructed capillary arc tubes may be operated at pressures of a hundred atmospheres or more[4] if a water jacket is provided to keep the quartz tube from reaching an excessively high temperature. Table III gives the operating characteristics and the spectral distribution of the visible radiation for an experimental tube of this type. Fig. 1 gives the spectral distribution of the energy radiated in a direction perpendicular to the arc axis for a similar tube[5] with an arc length of 1.0 cm. This tube was operated at 500 volts and 1.14 amperes (485 watts; pressure about 140 atmospheres). The spectral distribution given by Elenbaas[6] for a tube operated at a pressure of 130 atmospheres probably would be somewhat similar if it were expressed in terms of a normal spectrum, with the appropriate corrections for the dispersion of the spectrometer used for the measurements. However, Elenbaas obtained much more infrared of wave-lengths greater than 1.0μ because the layer of water traversed by the radiation was only 0.2 to 0.4 mm thick in the case of his measurements, but was about 12 mm thick when the data of Fig. 1 were obtained.

The peaks on Fig. 1 which represent the principal spectral lines are 40 to 100 angstrom units wide[7] at a point midway between the base and the tip of the peak. Only a small part of this breadth is instrumental because the peaks were determined with an exit slit width equivalent to about 2A at λ3654, 4A at λ4358, 7A at λ5461 and 26A at λ10,140. (For the rest of the spectral distribution measurements, the spectral range corresponding to the exit slit width was approx-

[5] We are indebted to Dr. N. T. Gordon of the General Electric Research Laboratories at Schenectady, New York, for supplying us with this tube.
[6] W. Elenbaas, Physica **3**, 859 (1936).
[7] The contours of these broadened lines will be reported and discussed in a later paper.

TABLE III. *Operating characteristics and spectral distribution of visible radiation for a water-cooled capillary arc tube.*

Inside Tube Diam. (cm)	Arc Length (cm)	Vapor Pressure (atm)	Volts	Amp.	Watts*	Lumens	L/W	Candles per mm²†
.2	1.2	110	505	1.35	580	38,200	66	445

Band of Wave-lengths	Intensities‡	Band of Wave-lengths	Intensities‡
7250–6960	42	5000–4890	46
6960–6705	41	4890–4790	57
6705–6470	38	4790–4695	56
6470–6245	40	4695–4605	52
6245–6045	45	4605–4515	62
6045–5865	67	4515–4430	90
5865–5690	125	4395–4320	150
5690–5535	114	4280–4210	68
5535–5390	197	4210–4140	62
5390–5250	47	4080–4020	105
5250–5123	34	4020–3960	42
5123–5000	35	3960–3900	38

* For arc tube only.
† Maximum value at center of arc.
‡ μw per cm² at one meter from arc axis.

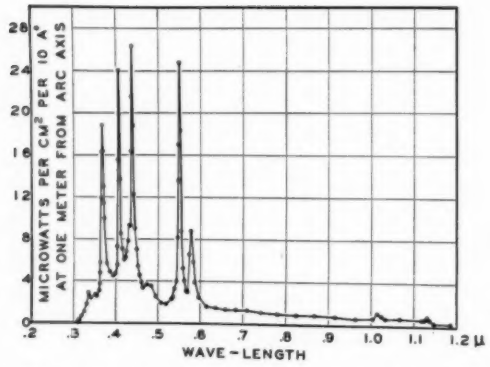


FIG. 1. Spectral distribution of radiation from water-cooled capillary arc tube operated at a pressure of about 140 atmospheres. Arc gap 10 mm; lamp at 500 volts, 1.13 amp., 485 watts.

imately equal to the extent of the spectral region between successive points on the graph.) The entrance slit was somewhat narrower than the exit slit in all cases.

The lines in the spectrum of a water-cooled capillary arc are broadened to such an extent that a separation of the spectrum into lines and continuous radiation would have little meaning. Nevertheless, a comparison of the spectral distribution given by Fig. 1 with the data in Table I for the type H-3 lamp shows that the water-cooled 140 atmosphere arc has a much greater proportion of its visible radiation in the continuous spectrum than does the air-cooled 28.5 atmosphere arc.

From the data in Tables I and III, the amount of light in any particular part of the spectrum of these different mercury arcs may be calculated. Table IV shows the percentage of the light from the H-1, H-2, and H-3 lamps, from the water-cooled capillary arc—and, for comparison, from the sun and the 500 watt, 115 volt tungsten lamp—which is in the wave-length region 6000 to 7600A. It is evident that the fraction of the light from a mercury arc which is in the red end of the spectrum increases with the vapor pressure of the mercury.[8] Nevertheless, even for the water-cooled capillary arc tube, the proportion of the light which is of wave-lengths greater than 6000A is much less than for sunlight.

TABLE IV. *Percentage of total light from mercury arcs operating at various pressures, from sun and from a tungsten lamp which is of wave-lengths 6000–7600A.*

SOURCE	MERCURY PRESSURE (ATM.)	PERCENT OF LIGHT 6000–7600A
H-1 lamp	1.4	1.0
H-2 lamp	.4	.6
H-3 lamp	28.5	1.9
Water-cooled capillary arc	110	7.0
Sun		20.0
500 watt, 115 volt tungsten lamp		27.0

The brightness at the center of the arc for the water-cooled tube is nearly eight times that of melting tungsten. This fact, coupled with the high luminous efficiency and the unusual spectral distribution of the radiation, has stimulated the study of these arcs. Tubes have been operated at still higher pressure, giving even greater brightness.[4] These tubes are still in the stage of laboratory curiosities; eventually they may find a limited commercial use in spite of the combined handicaps of water-cooling, high lamp voltage and relatively short life.

[8] See also J. W. Marden, N. C. Beese and George Meister, Elec. Eng. **55**, 1186 (1936).

Optical Changes in Freshly Evaporated Aluminum-Magnesium Films

HIRAM W. EDWARDS AND ROBERT P. PETERSEN
University of California at Los Angeles

(Received October 2, 1936)

This paper deals with experimental observations of the reflection and transmission characteristics of freshly evaporated films of an Al-Mg alloy (Al 95 percent, Mg 5 percent). Appropriate apparatus was arranged inside of an evaporation chamber so that measurements could be made during and after the deposition of the metallic film. Three types of changes in the reflection and transmission coefficients for white light were observed to take place. (1) The first type (occurring at pressures of 10^{-4} mm or less) consists of a small rapid decrease in reflectivity and increase in transmissivity. No further effect was observed to take place in the vacuum. (2) The second type occurs during the admission of air or other gases into the vacuum chamber. Similar to the first change, this consists of a decrease in reflectivity and an increase in transmissivity. (3) The third type takes place very slowly and over a period of several days. The final value of the reflection or transmission coefficient is approached asymptotically. In a typical case the magnitudes of the three types of changes for a film having an initial reflection coefficient of 58 percent were observed to be 8 percent, 5 percent, and 10 percent, respectively; and for another film having an initial transmission of 35 percent were 5 percent, 8 percent and 5 percent.

DURING the last three or four years considerable interest has been shown in the commercial development of color photography. The cameras for this purpose are designed to make two, or more frequently three negatives simultaneously. The light beam, coming in through the lens, is split into two or three parts by suitable semi-transparent mirrors properly positioned, and forms images upon photographic plates through selected filters.

Up to about a year ago the all important semi-transparent mirrors were made by depositing a thin silver film of appropriate thickness upon optically flat glass. Usually the silver films were covered with a protective layer of some colorless lacquer. In spite of precautionary measures the length of life of mirrors so protected was not long.

The newly developed process for making mirrors by evaporation together with the use of aluminum or an aluminum magnesium[1] alloy presented a product which was better suited for use in the color camera. The aluminum magnesium alloy seems to be superior to pure aluminum as a reflecting medium for semi-transparent mirrors primarily because of the greater hardness obtained. Several metals, when added to aluminum produce a definite increase in hardness, but magnesium appears to give a smaller absorption coefficient and therefore an increase in efficiency. With the semi-transparent aluminum magnesium films deposited upon white glass the absorption coefficient (including absorption by the glass) ranges from approximately 25 percent in the case of very thin films to less than 10 percent for nearly opaque films.

The writers have developed a technique for producing semi-transparent films of the aluminum magnesium alloy on glass up to an 8 x 10 inch size with a variation of less than 1 percent in the transmissivity over the entire plate. Because it requires only a very few seconds (in some cases as little as 1.2 seconds) to evaporate the desired amount of metal, considerable difficulty was experienced in obtaining a deposit of specified transmissivity.

Further difficulty lies in the fact that after the film has been deposited, its reflectivity and transmissivity change considerably within a short period of time. Because of these facts it seemed desirable to study these changes quantitatively.

An evaporation chamber having a volume of about 50 liters was used with the necessary appurtenances for evaporating an alloy consisting of aluminum 95 percent and magnesium 5 percent. A simple optical system, consisting of a six-volt incandescent lamp providing white light, a converging lens to collimate the light beam and a photoelectric cell, was arranged in the vacuum chamber so that intensities of the light reflected from either first or second surface or of the light transmitted could be measured. The evaporated alloy was condensed upon a glass support. Ob-

[1] Edwards, Phys. Rev. **43**, 205 (1933).

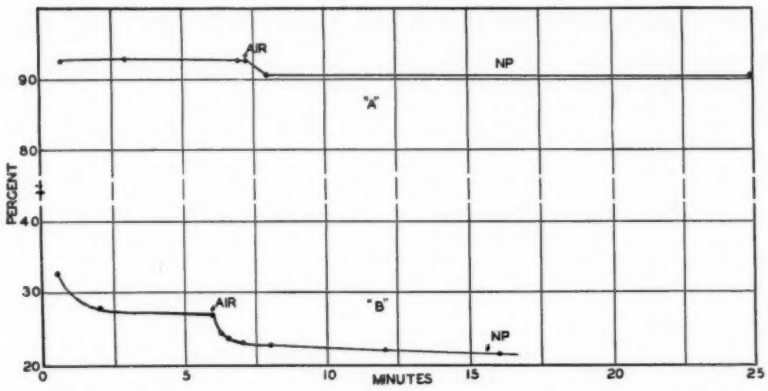


FIG. 1. Curves showing the changes in first surface reflection of evaporated films with time.

servations of the reflected or transmitted light made during evaporation were not accurate because of scattered light from the tungsten wires used to heat the alloy and are therefore not included in the results given below.

EXPERIMENTAL RESULTS

In all of the records made, the reflection or transmission characteristics of the evaporated film were recorded as a function of the time which was usually taken from the instant at which the evaporation was started. The time required for the evaporation varied from a few seconds for the thinnest films to a minute or two for the thickest films, the thicker films being very nearly opaque.

First surface reflection

In Fig. 1 are shown two typical records, one (*A*) for a nearly opaque film and the other (*B*) for a thin film. The vacuum (pressure less than 10^{-4} mm) was maintained for seven and six minutes, respectively. Air was admitted to the chamber at the time marked "Air" on the records and normal pressure was reached several minutes later (marked *NP* on the curves). All films which reflected 90 percent or less of the incident beam showed a marked decrease in reflectivity during the first minute or two after evaporation and thereafter changed unappreciably as long as the vacuum was maintained.

A second decrease in reflectivity occurred when air was admitted to the chamber. Here also the change was very rapid at first. The rate of decrease diminished rapidly, particularly after normal atmosphere pressure was reached.

A third change in the reflectivity was observed to take place. This is not shown in the figure. It was very small and took several days to complete.

Second surface reflection

In Fig. 2 three typical records are shown. The curves marked *A* and *C* represent the variations in the second surface reflections in all but two of the records made. Curve *B* is one of these two. Vacuum was maintained for ten minutes for all three records and normal atmospheric pressure reached in two or three minutes later. With more nearly opaque films than show in the figure, the variation in second surface reflectivity was very small.

Curve *B* shows what may be a spurious phenomenon. We cannot be sure why it should show a reversal of the seemingly orthodox behavior. It is possible that the metallic film was not in exact contact with the glass support and in consequence of the separation, an interference might have occurred. In such a case, a subsequent coalescence between film and glass support would show an increase in reflectivity rather than a decrease.

Transmission

Typical changes in transmissivity are shown in Fig. 3. These curves clearly show the characteristics of the phenomena and little comment need be made. Attention should be called, how-

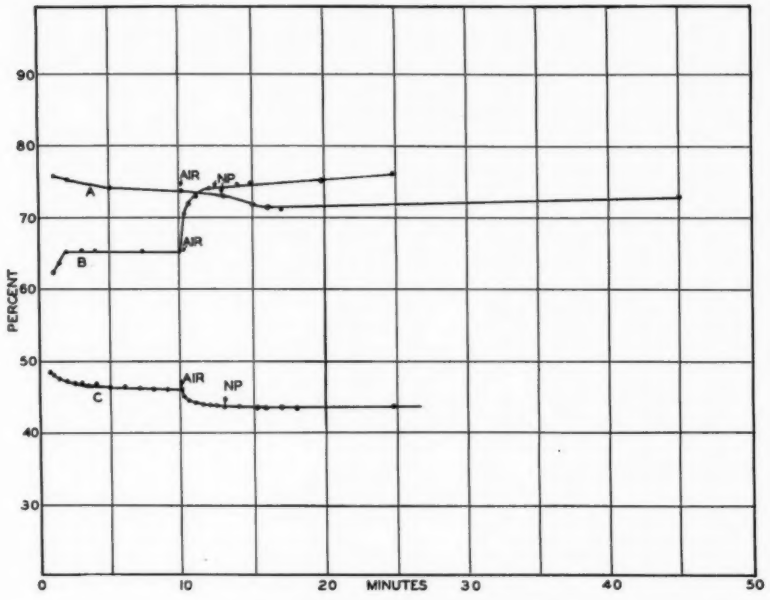


FIG. 2. Curves showing the changes in second surface reflection of evaporated films with time.

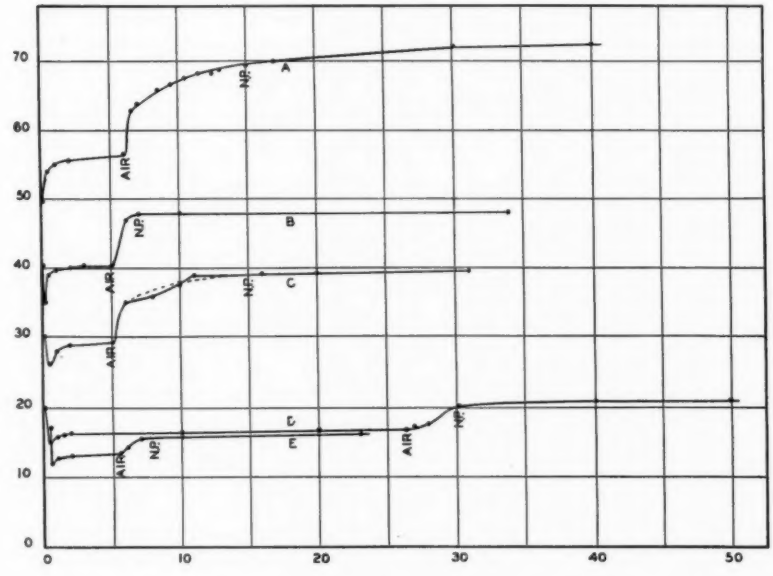


FIG. 3. Curves showing the changes in the transmission of evaporated films with time.

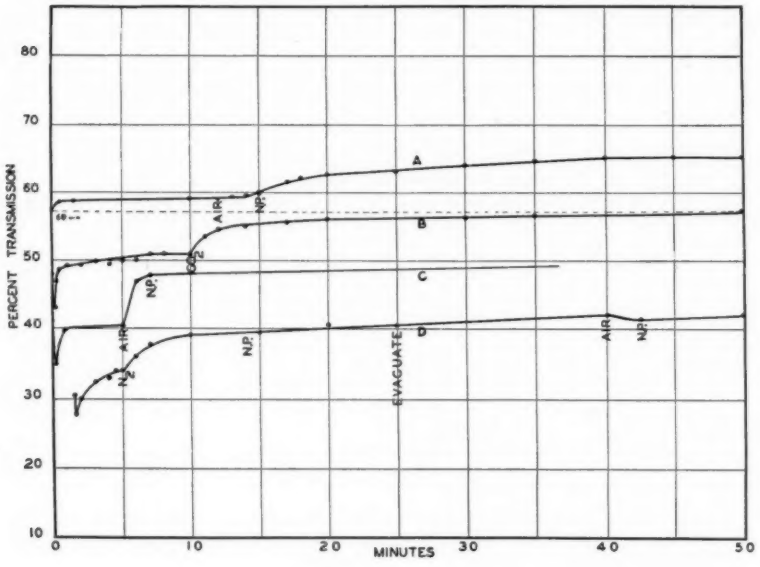


FIG. 4. Curves showing the changes of transmission with time when other gases, as indicated, are in contact with the evaporated film.

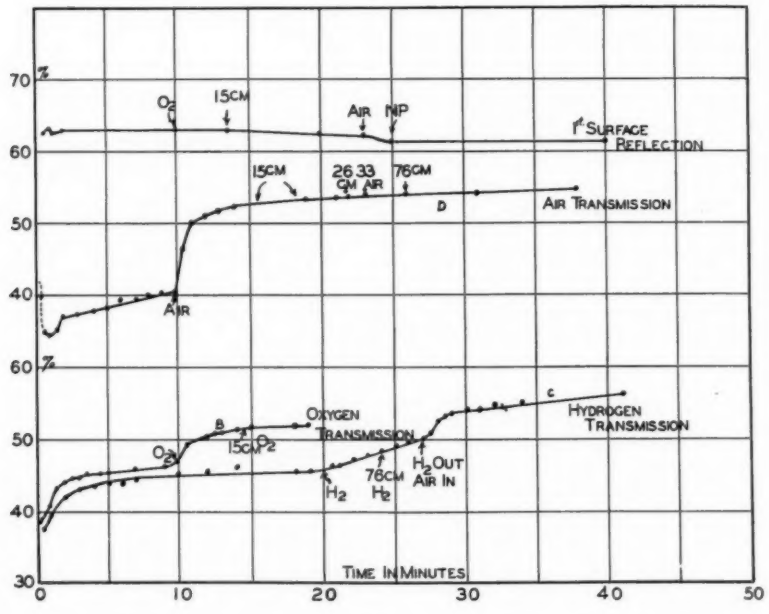


FIG. 5. Curves showing the changes in reflection and transmission in other gases with time.

ever, to the fact that the rate of change of transmissivity upon admitting air seems to be dependent upon the rate of increase of pressure in the vacuum chamber.

Obviously the causes which produce such changes in the reflection and transmission characteristics of thin Al-Mg films is of much interest and knowledge of the phenomena may be of considerable use. The remarks which follow are very largely of a speculative nature but there is some factual basis for them.

The first change (that which occurs in vacuum) may be accounted for on the basis of cohesive forces. When first deposited the metal must probably be in an amorphous form. Cohesive forces are large enough to produce some change in the amorphous material. The process of crystallization may be coexistent. Microscopic studies reported in the literature show definite formations of "aggregates," that is, scattered and irregular patches of more concentrated amounts of metal with spaces more or less empty of materials interspersed. Whatever be the cause, it is nevertheless remarkable that the change goes on so quickly.

The second change is probably due in part to an increase of pressure and, in part, to oxidation.

To determine whether pressure had any real effect in the amount of the second change, other gases were admitted to the chamber in place of air. The results are shown in Fig. 4. Nitrogen was first used (curve *D*) with dubious results, however, since it was found later to contain some oxygen. Carbon dioxide (curve *C*) was next used but here again a possible effect due to pressure alone may be obscured by the possibility that either Al or Mg in the existent form might readily reduce the carbon dioxide forming the corresponding oxides.

Pure oxygen (Fig. 5, curve *B*) was next tried but the results here did not indicate any increased rate of change as might have been expected. At any rate, the experiment proved to be too dangerous for in one trial an explosion occurred from a rapid oxidation of the oil in the diffusion pumps.

In curve *D*, Fig. 5, partial pressures of air were tried. This test showed that if pressure plays a part, only a small pressure is necessary.

The last gas tried was hydrogen (curve *C*, Fig. 5). When hydrogen was introduced there was a small increase in the slope of the curve which may be explained as due to pressure. A few minutes after the pressure of the hydrogen had reached atmospheric, the door to the vacuum chamber was removed and air allowed to enter, forcing the hydrogen out. Here the rapid increase in the transmission is clearly due to oxidation.

The third change is without doubt due to a combination of oxidation in the more protected parts of the film with crystallization. The fact that the hardness of the surface increases slowly for several weeks corroborates this latter statement.

Absence of Fresnel's "Most Useful Effect" in the Hyperbolic Reflector

L. E. Dodd
Department of Physics, University of California at Los Angeles
(Received November 20, 1936)

In Fresnel's analysis of the parabolic reflector, considering finite size of source he defined a quantity he called the useful effect. The present paper applies Fresnel's test to the hyperbolic reflector. It is shown mathematically that for the hyperbola the useful effect reaches no maximum except for degenerate cases.

IN Fresnel's analysis of the parabolic reflector with source of finite size, a quantity is defined called the useful effect (*l'effet utile*).[1] His general problem connected with lighthouses was to send a beam as far as possible. More specifically the problem, as far as concerned performance of the reflector itself, was to get the most light projected with least dispersion due to finite size of source.

For a given point on the reflecting surface, the amount of light projected from a differential zone containing this point, is proportional to the differential solid angle between two geometrical cones that bound the zone, their common vertex being in the source and on the axis of the conic. Furthermore, for this point on the reflector the dispersion of the extreme rays is less the greater the distance from the source. This distance was therefore combined by Fresnel with the differential solid angle, in a quantity proportional to both of these. Integration then gave the useful effect of the mirror as a whole, a function of the parameter of the parabola and the angular opening of the mirror subtended at the focal point. When this function is a maximum there is present the "most useful effect."

Fresnel solved this problem in the parabola for two practical cases: (1) constant area of mirror surface, and (2) constant aperture; i.e., linear diameter of mirror opening. He thus limited the mirror in these two different ways, before proceeding to determine the maximum value of useful effect in each case. In case (1) the parameter of the conic was first replaced by its equivalent function of the area and the angular opening as defined above. In case (2), correspondingly, the parameter was replaced by its equivalent function of the aperture diameter and the angular opening. In both cases the replacement function varied only with the angular opening.

The practical questions to be answered in the two problems were, respectively: (1) For a given area of mirror surface, what is the best parabola (i.e., the best focal length, or in Fresnel's words "degree of curvature") and the best angular opening? (2) For a given diameter of mirror opening, what is the best parabola and the best angular opening (as Fresnel put it, the best "depth")? In each case the size of angular opening was first found that gave the most useful effect. This in turn, when combined with the chosen mirror area or mirror aperture, determined the parabola. Fresnel checked both his solutions by numerical calculations, finding it necessary in both cases to make successive approximations.

Using the definition of useful effect as given above, one may follow a generally similar procedure with the hyperbola. Perhaps having no occasion to do so, Fresnel did not apply his test to this curve, as far as the writer knows. The two parameters seemed to promise at the outset algebraic complications. The problem however is of some interest and value in the practical projection of light.

A standard equation for the hyperbola is

$$x^2/a^2-y^2/b^2=1. \tag{1}$$

The eccentricity is given by $e^2=(a^2+b^2)/a^2$.

The parametric equations for point P on the curve (see Fig. 1) are

$$x=c-r\cos\alpha, \quad y=r\sin\alpha, \tag{2}$$

where $c=ae=(a^2+b^2)^{\frac{1}{2}}$, and α is the angular opening with respect to the focus. From (1) and (2),

[1] Fresnel, D'Augustin, *Oeuvres complètes*, Vol. 3 (Paris, 1870), p. 41.

$r=b^2((a^2+b^2)^{\frac{1}{2}}\cos\alpha(\pm)a)/(b^2\cos^2\alpha-a^2\sin^2\alpha)$

and only the negative sign is retained, so that, simplifying,

$$r=b^2/(a+c\cos\alpha). \tag{3}$$

Now, letting λ be the linear diameter of the aperture, taken constant, we have

$$\lambda/2=r\sin\alpha$$
$$=(c^2-a^2)\sin\alpha/(a+c\cos\alpha)\equiv B. \tag{4}$$

The useful effect is

$$E=2\pi\int_0^{\alpha} r\sin\alpha\,d\alpha=2\pi b^2\int\frac{\sin\alpha\,d\alpha}{a+c\cos\alpha}$$
$$=\frac{-2\pi b^2}{c}\int\frac{-c\sin\alpha\,d\alpha}{a+c\cos\alpha}.$$

This may be integrated at once to yield $E=-A\log(a+c\cos\alpha)+K$, where $A=2\pi b^2/c$ and K is the integration constant. Since E must be zero when $\alpha=0$, we have $K=A\log(a+c)$, whence[2]

$$E=2\pi\left(\frac{c^2-a^2}{c}\right)\log\left(\frac{a+c}{a+c\cos\alpha}\right). \tag{5}$$

Now Eq. (5) and the constraint applied by Eq. (4) present together an isoperimetric problem in maxima and minima. Write p for the quantity $(c^2-a^2)/c$. Since the eccentricity $e=c/a$, we have

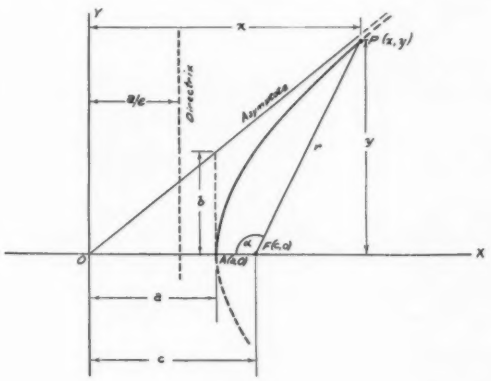


FIG. 1.

[2] In the parabola the constant of integration at this point comes out zero.

$(c-a)=ep/(1+e)$, whence from (4),

$$p=B(1+e\cos\alpha)/(e\sin\alpha),$$

and Eq. (5) then becomes

$$\frac{E}{2\pi B}=\frac{(1+e\cos\alpha)}{e\sin\alpha}\log\left(\frac{1+e}{1+e\cos\alpha}\right)\equiv w. \tag{6}$$

Since λ is an arbitrary constant, we now have an expression for the useful effect as a function of two independent variables; *viz.*, the ratio of the two hyperbolic parameters, and the angular opening. An equation is desired for α making E a maximum. Clearly, the maxima and minima of E and w occur for the same values of α and e. Such maxima and minima will be found among the simultaneous solutions of the two equations: $\partial w/\partial e=0$, and $\partial w/\partial\alpha=0$. We proceed to get these equations and to solve them.

$$\frac{\partial w}{\partial e}=\frac{1}{e\sin\alpha}\left[\frac{(1-\cos\alpha)}{(1+e)}+\frac{1}{e}\log\left(\frac{1+e\cos\alpha}{1+e}\right)\right], \tag{7}$$

$$\frac{\partial w}{\partial\alpha}=1+\left(\frac{e+\cos\alpha}{e\sin^2\alpha}\right)\log\left(\frac{1+e\cos\alpha}{1+e}\right). \tag{8}$$

Equating (7) and (8) to zero gives, respectively,

$$\frac{e(1-\cos\alpha)}{(1+e)}=\log\left(\frac{1+e}{1+e\cos\alpha}\right), \tag{9}$$

$$\frac{e\sin^2\alpha}{e+\cos\alpha}=\log\left(\frac{1+e}{1+e\cos\alpha}\right). \tag{10}$$

Solving (9) and (10) simultaneously gives

$$(1-\cos\alpha)/(1+e)=\sin^2\alpha/(e+\cos\alpha), \tag{11}$$

solutions of which are

(a) $\cos\alpha=+1$, $\alpha=0$, which means no mirror,

(b) $\cos\alpha=-e$, giving an imaginary situation in the hyperbola, the marginal radius vector from focus to curve not having a slope steeper than that of the asymptote, a condition necessary for a finite mirror,

(c) $\cos\alpha=-1/e$, requiring the margin of the mirror to be at an infinite distance, angle α approaching 180°.

Thus having found the possible mathematical solutions of Eq. (11), all of which require degener-

ate cases in the hyperbola, we can conclude that no hyperbolic mirror can be designed that will give Fresnel's most useful effect.[3]

[3] Since no solution is possible with a hyperbolic mirror of any assigned finite opening, the reflecting area being variable, it is obviously unnecessary to solve for the case of constant finite area, the mirror opening being variable.

In conclusion the writer acknowledges helpful suggestions from Professor W. M. Whyburn in the mathematical analysis. He would mention also the firm of Bardwell and McAlister, Inc., Los Angeles, who sponsored investigational work in which this problem was met and solved.

A Polarization Phenomenon in the Hilger E_1 Quartz Spectrograph

JOSEPH W. ELLIS AND JOSEPH KAPLAN
University of California at Los Angeles
(Received November 23, 1936)

A YEAR ago we presented an analysis[1] of the numerous maxima and minima of intensity which usually occur in a continuous spectrum produced by a Hilger E_1 spectrograph when plane polarized light is used. Our analysis of the observed phenomena, based upon the action of the reflecting quartz prism as a thick wave-plate, the rotation of the plane of polarization by the quartz lens and the analyzer action of the incident face of the dispersing prism, was satisfactory with one exception. This exception was, as we then pointed out, our failure to explain why the visibility minima for these intensity maxima and minima do not occur at wave-length intervals corresponding to rotations by the lens of the plane of polarization of amounts equal to integer multiples of 90°. This apparent anomaly is made clear in a recent paper by Bolla,[2] in which he independently arrives at the main conclusions of our paper. We had assumed that the variation in path lengths through the dispersing prism would so distribute the azimuths of vibration about the plane of incidence that no further analyzer action would occur at the emergent face. This seems not to be the case. Since in this autocollimation spectrograph none of the waves goes through at minimum deviation, there is a resultant (partially counteracting) rotation of the plane of polarization in the prism. This rotation, the rotation by the lens and the analytical action of the emergent face, if acting alone, would yield a set of visibility minima whose wave-length intervals would be slightly larger than those produced by the first face. The combined action of the two faces would yield intervals intermediate between the two independent ones. These intervals would represent nearly integer multiples of a rotation angle less than 90° on the basis of rotation by the lens alone.

It is a further analyzer action by the face of the photographic plate that gives the intervals outlined above alternately high and low values. Spacings due to this action alone would be very large because of the rotational compensation by the lens as the rays return through it. The combined effects of the three analyzer actions cause alternate visibility minima to appear to occur at positions corresponding to integer multiples of 90°, an effect more fortuitous than significant.

[1] Ellis and Kaplan, J.O.S.A. **25**, 357 (1935).
[2] G. Bolla, Nuovo Cimento **13**, 241 (1936).